

MAT4450 - Spring 2025 - Solutions of exercises - Set 1

Exercise 1. Let X be a topological space which is first countable, that is, every $x \in X$ has a countable neighborhood basis. Let $A \subseteq X$ and $x \in X$. Then we have that

$$x \in \overline{A} \Leftrightarrow \text{there exists a sequence } \{x_n\} \text{ in } A \text{ such that } x_n \rightarrow x \text{ as } n \rightarrow \infty.$$

We know that in any topological space the implication $(\Leftarrow)$ holds for a net, hence for a sequence. So it suffices to show that $(\Rightarrow)$ holds. Assume that $x \in \overline{A}$. Let $\{B_n : n \in \mathbb{N}\}$ denote a neighborhood basis at x . By replacing each B_n by $\bigcap_{j=1}^n B_j$ if necessary, we may assume that $B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we have that $B_n \cap A \neq \emptyset$, so we may pick $x_n \in B_n \cap A$. Consider now any $V \in \mathcal{N}_x$. Choosing n_0 such that $B_{n_0} \subseteq V$, we get that $x_n \in B_n \subseteq B_{n_0} \subseteq V$, hence $x_n \in V$, for every $n \geq n_0$. Thus, $x_n \rightarrow x$ as $n \rightarrow \infty$, as desired.

Exercise 2. Let $X = \mathcal{F}(\mathbb{R}, \mathbb{R})$ consists of all real-valued functions on $\mathbb{R}$ and equip it with the topology of pointwise convergence. We recall that a neighborhood basis $\mathcal{B}_f$ at $f \in X$ is given by

$$\mathcal{B}_f = \{V_{f,S,\varepsilon} : S \text{ is a finite subset of } \mathbb{R}, \varepsilon > 0\},$$

where $V_{f,S,\varepsilon} := \{g \in X : |f(s) - g(s)| < \varepsilon \text{ for every } s \in S\}$.

For $h \in X$, set $\text{supp}(h) := \{t \in \mathbb{R} : h(t) \neq 0\}$. Moreover, set $A := \{h \in X : \text{supp}(h) \text{ is finite}\}$.

a) We show that $\overline{A} = X$ in two different ways:

i) Let $f \in X$. Consider $V \in \mathcal{N}_f$. We may then find some $V_{f,S,\varepsilon} \in \mathcal{B}_f$ such that $V_{f,S,\varepsilon} \subseteq V$. Define $f_S \in A$ by $f_S(t) = f(t)$ when $t \in S$, and $f_S(t) = 0$ when $t \notin S$. Then we clearly get that $f_S \in (V_{f,S,\varepsilon} \cap A) \subseteq (V \cap A)$, so $V \cap A \neq \emptyset$. Thus, $f \in \overline{A}$.

ii) Let $f \in X$. For every finite subset S of X , i.e., $S \in \mathcal{P}_{\text{fin}}(X)$, define f_S as above. Considering $\Lambda := \mathcal{P}_{\text{fin}}(X)$ as a directed set w.r.t. inclusion, we get a net $\{f_S\}_{S \in \Lambda}$ in A , which converges to f :

Indeed, let $V \in \mathcal{N}_f$. Choose $V_{f,S,\varepsilon} \in \mathcal{B}_f$ such that $V_{f,S,\varepsilon} \subseteq V$. Then for every $S' \in \Lambda$ such that $S \subseteq S'$ we have that $f_{S'} \in V_{f,S,\varepsilon} \subseteq V$ (since $|f(s) - f_{S'}(s)| = |f(s) - f(s)| = 0 < \varepsilon$ for all $s \in S$). Thus, again, $f \in \overline{A}$.

b) Let f be any function in $X = \overline{A}$ such that $\text{supp}(f)$ is uncountable. Then there is no sequence in A which converges to f .

Assume (for contradiction) that there exists a sequence $\{f_n\}$ in A such that $f_n \rightarrow f$ as $n \rightarrow \infty$. Set $T := \bigcup_{n \in \mathbb{N}} \text{supp}(f_n)$. Then T is countable. Moreover, $\text{supp}(f) \subseteq T$. Indeed, let $x \in \text{supp}(f)$. We may then find $n \in \mathbb{N}$ such that $|f(x) - f_n(x)| < |f(x)|/2$. This implies that $|f_n(x)| > |f(x)|/2 > 0$. Thus, $x \in \text{supp}(f_n) \subseteq T$. Since T is countable, we get that $\text{supp}(f)$ is countable, contradicting the hypothesis.

Exercise 3. Let X be a nonempty set and $A \subseteq X$. Consider a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X . Then the following statements are true.

a) If $\{x_\alpha\}$ is eventually in A , then it is frequently in A :

Assume $\{x_\alpha\}$ is eventually in A , i.e., there exists $\alpha_0 \in \Lambda$ such that $x_\alpha \in A$ for all $\alpha \geq \alpha_0$. Let $\beta \in \Lambda$. We may then find $\gamma \in \Lambda$ such that $\gamma \geq \alpha_0$ and $\gamma \geq \beta$. This gives that $x_\gamma \in A$. Hence, $\{x_\alpha\}$ is frequently in A .

b) $\{x_\alpha\}$ is not frequently in A if and only if it is eventually in $X \setminus A$:

We have that $\{x_\alpha\}$ is not frequently in A if and only if there exists some $\alpha_0 \in \Lambda$ such that $x_\beta \notin A$ for all $\beta \geq \alpha_0$, if and only if $\{x_\alpha\}$ is eventually in $X \setminus A$.

c) If $\{x_\alpha\}$ is eventually in A , then it is not eventually in $X \setminus A$, and if $\{x_\alpha\}$ is eventually in $X \setminus A$, then it is not eventually in A :

If $\{x_\alpha\}$ is eventually in $X \setminus A$, then it follows from b) that $\{x_\alpha\}$ is not frequently in A , and then a) gives that $\{x_\alpha\}$ is not eventually in A . This shows that the second assertion holds for every subset A of X . Applying this assertion to the subset $X \setminus A$, we get that the first assertion holds too.

Exercise 4. *Let X be a topological space. Then X is Hausdorff if and only if every net in X converges to at most one point.*

This is actually Proposition 1.5.2 in Pedersen's book (see p. 23, where you will also find the proof).

About **Exercise 1.3.3** in Pedersen's book.

For each λ let us denote by x_λ the sum defined on the left (using the inf) and by y_λ the sum defined on the right (using the sup). Then we clearly have that

$$m(b-a) \leq x_\lambda \leq y_\lambda \leq M(b-a),$$

where $m := \inf\{f(x) : a \leq x \leq b\}$, $M := \sup\{f(x) : a \leq x \leq b\}$. Moreover, one checks that the net $\{x_\lambda\}$ is non-decreasing, while $\{y_\lambda\}$ is non-increasing. It follows that both these nets are convergent, with

$$m(b-a) \leq \lim_{\lambda} x_\lambda \leq \lim_{\lambda} y_\lambda \leq M(b-a).$$

Now, if one says that the function f is R-integrable when $\lim_{\lambda} x_\lambda = \lim_{\lambda} y_\lambda$, then one has to work a bit to show that this is equivalent to f being Riemann-integrable. As this is somewhat outside the scope of this course, we skip the details (some of which can be found in Enstad's notes).