

MAT4450 - Spring 2026 - Solutions of exercises - Set 10

NB: Exercices 45 and 47 are part of the compulsory assignment.

Exercise 46

Consider the unital commutative Banach algebra $\mathcal{A} = \ell^1(\mathbb{Z}, \mathbb{C})$ (w.r.t. the $\|\cdot\|_1$ -norm). Set $\mathcal{B} := \{f \in \mathcal{A} : f(n) = 0 \text{ for all } n < 0\}$.

a) $\mathcal{B}$ is a norm-closed subalgebra of $\mathcal{A}$ which contains the unit of $\mathcal{A}$ (hence $\mathcal{B}$ is also a unital commutative Banach algebra):

It is immediate that $\mathcal{B}$ is a subspace of $\mathbb{C}$. Recall that for each $k \in \mathbb{Z}$, $\delta_k \in \mathcal{A}$ is defined by $\delta_k(j) = 1$ if $j = k$, while $\delta_k(j) = 0$ otherwise. Then it is clear that $f = \sum_{k \in \mathbb{Z}} f(k)\delta_k$ (w.r.t. $\|\cdot\|_1$) for all $f \in \mathcal{A}$.

Hence, if $f \in \mathcal{B}$, we get that $f = \sum_{k=0}^{\infty} f(k)\delta_k$ (w.r.t. $\|\cdot\|_1$). It follows readily that

$$\mathcal{B} = \overline{\text{span}\{\delta_k : k \in \mathbb{N} \cup \{0\}\}}^{\|\cdot\|_1},$$

so $\mathcal{B}$ is closed in $\mathcal{A}$. Moreover, as $\delta_k * \delta_l = \delta_{k+l}$ for all k, l , it is also straightforward to deduce that $\mathcal{B}$ is closed under the convolution product. Finally, the unit of $\mathcal{A}$ is δ_0 , which belongs to $\mathcal{B}$.

b) Let $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. Then the map $\gamma_\lambda : \mathcal{B} \rightarrow \mathbb{C}$ given by

$$\gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n$$

for all $f \in \mathcal{B}$ is well-defined and belongs to $\widehat{\mathcal{B}}$:

Let $f \in \mathcal{B}$. Since $\sum_{n=0}^{\infty} |f(n)\lambda^n| \leq \sum_{n=0}^{\infty} |f(n)| = \|f\|_1 < \infty$, the complex series $\sum_{n=0}^{\infty} f(n)\lambda^n$ is absolutely convergent, hence convergent. Thus $\gamma_\lambda(f)$ makes sense, so we get that $\gamma_\lambda : \mathcal{B} \rightarrow \mathbb{C}$ is well-defined.

We have now to check that γ_λ is linear, bounded, multiplicative and nonzero. Linearity is easy. Since

$$|\gamma_\lambda(f)| \leq \sum_{n=0}^{\infty} |f(n)| = \|f\|_1$$

for all $f \in \mathcal{B}$, γ_λ is bounded. To show multiplicativity, consider first $k, l \in \mathbb{N} \cup \{0\}$. Then

$$\gamma_\lambda(\delta_k * \delta_l) = \gamma_\lambda(\delta_{k+l}) = \lambda^{k+l} = \lambda^k \lambda^l = \gamma_\lambda(\delta_k) \gamma_\lambda(\delta_l).$$

It follows that γ_λ is multiplicative on the subalgebra $\mathcal{B}_0 := \text{span}\{\delta_k : k \in \mathbb{N}\}$. By density of $\mathcal{B}_0$ in $\mathcal{B}$ and continuity of γ_λ , we get that γ_λ is multiplicative on $\mathcal{B}$. Finally, $\gamma_\lambda(\delta_0) = \lambda^0 = 1$, so $\gamma_\lambda \neq 0$.

c) The map $\lambda \mapsto \gamma_\lambda$ is a homeomorphism from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$:

• Injectivity: Assume $\gamma_\lambda = \gamma_\mu$ for $\lambda, \mu \in \mathbb{D}$. Then $\lambda = \gamma_\lambda(\delta_1) = \gamma_\mu(\delta_1) = \mu$.

• Surjectivity: Let $\gamma \in \widehat{\mathcal{B}}$. Set $\lambda = \gamma(\delta_1) \in \mathbb{C}$. Then $|\lambda| = |\gamma(\delta_1)| \leq \|\gamma\| \|\delta_1\|_1 = 1$. So $\lambda \in \mathbb{D}$. Further, since γ is continuous and $\delta_n = (\delta_1)^n$ for all n , we get that

$$\gamma(f) = \sum_{n=0}^{\infty} f(n)\gamma(\delta_n) = \sum_{n=0}^{\infty} f(n)\gamma(\delta_1)^n = \sum_{n=0}^{\infty} f(n)\lambda^n = \gamma_\lambda(f)$$

for all $f \in \mathcal{B}$, so $\gamma = \gamma_\lambda$.

• We have shown that the map $\lambda \mapsto \gamma_\lambda$ is a bijection ϕ from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$, with inverse $\psi : \widehat{\mathcal{B}} \rightarrow \mathbb{D}$ given by $\gamma \mapsto \gamma(\delta_1)$. If $\{\gamma_\alpha\}$ is a net in $\widehat{\mathcal{B}}$ converging to $\gamma \in \widehat{\mathcal{B}}$, then $\psi(\gamma_\alpha) = \gamma_\alpha(\delta_1) \rightarrow \gamma(\delta_1) = \psi(\gamma)$. Thus, ψ is continuous. Since $\widehat{\mathcal{B}}$ is compact and $\mathbb{D}$ is Hausdorff, we get that $\phi = \psi^{-1}$ is continuous too. Hence ϕ is a homeomorphism, as desired.

Next, identifying $\widehat{\mathcal{B}}$ with $\mathbb{D}$, we get that the Gelfand transform of $\mathcal{B}$ is the map $\Gamma : \mathcal{B} \rightarrow C(\mathbb{D})$ given by

$$[\Gamma(f)](\lambda) = \sum_{n=0}^{\infty} f(n)\lambda^n \quad \text{for } f \in \mathcal{B} \text{ and } \lambda \in \mathbb{D}$$

Indeed, for all $f \in \mathcal{B}$ and $\lambda \in \mathbb{D}$, we get that

$$[\Gamma(f)](\gamma_\lambda) = \gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n.$$

Thus, by identifying λ with γ_λ , we obtain the formula above.

It follows that Γ is one-to-one: Indeed, if $\Gamma(f_1) = \Gamma(f_2)$ for $f_1, f_2 \in \mathcal{B}$, then we get that

$$\sum_{n=0}^{\infty} f_1(n)\lambda^n = \sum_{n=0}^{\infty} f_2(n)\lambda^n$$

for all $\lambda \in \mathbb{D}$. By the uniqueness of a power series expansion around zero, we get that $f_1(n) = f_2(n)$ for all $n \geq 0$, i.e., $f_1 = f_2$.

The range of Γ may be described as the set of continuous complex functions on $\mathbb{D}$ having an absolutely convergent power series expansion around zero valid in $\mathbb{D}$, i.e., which are analytic on the open disk $\mathbb{D}^\circ$. It follows that $\Gamma(\mathcal{B})$ is the so-called disk algebra $\mathcal{A}(\mathbb{D})$.