

MAT4450 - Spring 2026 - Solutions of exercises - Set 14

Exercise 57

Let H be a Hilbert space (over $\mathbb{C}$) and $T \in \mathcal{B}(H)$. Let $T = U|T|$ be the polar decomposition of T .

a) *The following relations hold:*

- $U^*U|T| = |T|$
- $U^*T = |T|$
- $UU^*T = T$
- $|T^*| = U|T|U^*$
- $T^* = U^*|T^*|$, and this gives the polar decomposition of T^* .

We have seen in the lecture that U^*U is the orthogonal projection of H onto $\overline{|T|(H)}$, while UU^* is the orthogonal projection of H onto $\overline{T(H)}$. This gives that

$$U^*U|T|\xi = |T|\xi \quad \text{and} \quad UU^*T\xi = T\xi$$

for all $\xi \in H$, hence that $U^*U|T| = |T|$ and $UU^*T = T$. This implies that $U^*T = U^*U|T| = |T|$. Moreover, since

$$|T^*|^2 = TT^* = U|T|(U|T|)^* = U|T||T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2,$$

and $U|T|U^* \geq 0$ (because $\langle U|T|U^*\xi, \xi \rangle = \langle |T|U^*\xi, U^*\xi \rangle \geq 0$ for all $\xi \in H$), we get that $|T^*| = U|T|U^*$ (by the uniqueness property of square roots).

Finally, using the identities we have shown above, we get

$$T^* = (U|T|)^* = |T|U^* = U^*U|T|U^* = U^*|T^*|.$$

Now, since U is a partial isometry, U^* is a partial isometry too. Further, since the final space of U is $\overline{T(H)}$, we have that $\ker(U^*) = U(H)^\perp = T(H)^\perp = \ker(T^*)$. So we can conclude that $T^* = U^*|T^*|$ is the polar decomposition of T^* (by the uniqueness property of polar decomposition).

b) Assume that T is invertible. Then U is unitary:

Since T is invertible, we have $\ker(T) = \{0\}$ and $T(H) = H$, hence $\overline{T(H)} = H$. So U^*U is the orthogonal projection of H onto

$$\overline{|T|(H)} = \ker(|T|)^\perp = \ker(T)^\perp = \{0\}^\perp = H,$$

that is, $U^*U = I_H$. Moreover, UU^* is the orthogonal projection of H onto $\overline{T(H)} = H$, i.e., $UU^* = I_H$. Thus, U is unitary.

Exercise 58

Let H be a Hilbert space (over $\mathbb{C}$) and let $T \in \mathcal{B}(H)$. We describe the polar decomposition of T in the following cases:

- $T \geq 0$

We then have $|T| = (T^2)^{1/2} = T$. Moreover, the partial isometry U restricts to the identity operator on $M := \overline{T(H)}$ and is zero on $M^\perp = \overline{T(H)}^\perp$. Hence U is the orthogonal projection P_M of H onto $M = \overline{T(H)}$. Hence the polar decomposition of T is $T = P_M T$.

- T is an orthogonal projection

This is a special case of the case above. We then get that $|T| = T$ and $U = T$.

- T is a partial isometry

We know then that T^*T is an orthogonal projection. So $|T| = (T^*T)^{1/2} = T^*T$. Moreover, T^*T is the orthogonal projection of H onto $\overline{T(H)}$. Thus, $T = T T^*T = T |T|$, and this is the polar decomposition of T (by the uniqueness property), i.e., $U = T$.

- T is an isometry

This is a special case of the case above. We then get that $U = T$ and $|T| = I_H$.

Exercise 59

Assume H is a separable infinite-dimensional Hilbert space (over $\mathbb{C}$), and let $\{e_j : j \in \mathbb{N}\}$ be an o.n.b. for H . Let $S \in \mathcal{B}(H)$ denote the (unilateral) shift-operator associated with this basis, so S is determined by $S(e_j) = e_{j+1}$ for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$. The following facts hold:

- S^n is an isometry with range $H_n := \{e_1, \dots, e_n\}^\perp$

It should be well-known that S is an isometry, but here is the argument. Let $\xi \in H$. Then

$$S(\xi) = S\left(\sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle e_j\right) = \sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle S(e_j) = \sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle e_{j+1}.$$

Since $\{e_{j+1}\}_{j \in \mathbb{N}}$ is an orthonormal set in H , we get that

$$\|S(\xi)\|^2 = \left\| \sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle e_{j+1} \right\|^2 = \sum_{j \in \mathbb{N}} |\langle \xi, e_j \rangle|^2 = \|\xi\|^2,$$

which proves the claim.

The product of two isometries is easily seen to be an isometry. It readily follows that S^n is an isometry.

Moreover, using the formula for $S(\xi)$ above, we deduce that

$$S^n(\xi) = \sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle e_{j+n}.$$

Thus, if H_n denotes the range of S^n , this implies that H_n is contained in $\overline{\text{span}\{e_{j+n}, j \in \mathbb{N}\}}$. On the other hand, since $e_{j+n} = S^n(e_j)$ for each $j \in \mathbb{N}$, we get that $\text{span}\{e_{j+n}, j \in \mathbb{N}\} \subseteq H_n$. As S^n is an isometry, its range H_n is complete, hence closed. Thus, we get that $\overline{\text{span}\{e_{j+n}, j \in \mathbb{N}\}} \subseteq H_n$. Altogether, we conclude that $H_n = \overline{\text{span}\{e_{j+n}, j \in \mathbb{N}\}}$.

As $\{e_1, \dots, e_n\}^\perp = \overline{\text{span}\{e_{j+n}, j \in \mathbb{N}\}}$, this shows that $H_n = \{e_1, \dots, e_n\}^\perp$, as desired.

- $(S^n)^* = (S^*)^n$ is a partial isometry with initial space H_n and final space H

The operator S^n , being an isometry, is a partial isometry with initial space H and final space H_n . Thus, as seen in a lecture, this implies that $(S^n)^*$ is a partial isometry with initial space H_n and final space H .