

MAT4450 - Spring 2026 - Solutions of exercises - Set 3

Exercise 9

Let X be a vector space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $\mathcal{S}$ be a nonempty family of seminorms on X . Let $\tau(\mathcal{S})$ denote the weak topology on X induced by $\mathcal{S}$. We recall that $\tau(\mathcal{S})$ is the weak topology on X determined by the family $\{\rho_{(\sigma,z)}\}_{(\sigma,z) \in \mathcal{S} \times X}$ of real-valued functions on X given by $\rho_{(\sigma,z)}(y) := \sigma(y - z)$ for all $y \in X$.

Let $x \in X$. For $\sigma \in \mathcal{S}$ and $\varepsilon > 0$, set $B_\varepsilon^\sigma(x) := \{y \in X : \sigma(y - x) < \varepsilon\}$. Also, if F is a finite nonempty subset of $\mathcal{S}$, set

$$B_\varepsilon^F(x) := \bigcap_{\sigma \in F} B_\varepsilon^\sigma(x) = \{y \in X : \sigma(y - x) < \varepsilon \text{ for all } \sigma \in F\}.$$

Moreover, for $z \in X$, set $V_{(\sigma,z),\varepsilon}(x) := \{y \in X : |\sigma(y - z) - \sigma(x - z)| < \varepsilon\}$.

a) Let $z \in X$ and $\varepsilon > 0$. Then $V_{(\sigma,x),\varepsilon}(x) = B_\varepsilon^\sigma(x) \subseteq V_{(\sigma,z),\varepsilon}(x)$:

The equality on the left follows immediately from the definitions. To show the inclusion on the right, let $y \in B_\varepsilon^\sigma(x)$. Then

$$|\sigma(y - z) - \sigma(x - z)| \leq \sigma((y - z) - (x - z)) = \sigma(x - y) < \varepsilon,$$

so $y \in V_{(\sigma,z),\varepsilon}(x)$, as desired.

b) The family $\mathcal{B}_x := \{B_\varepsilon^F(x) : F \text{ is a finite nonempty subset of } \mathcal{S} \text{ and } \varepsilon > 0\}$ is a neighborhood basis at x (for $\tau(\mathcal{S})$):

Using Exercise 8, we get that the family

$$\mathcal{V}_x = \left\{ \bigcap_{k=1}^n V_{(\sigma_k, z_k), \varepsilon}(x) : n \in \mathbb{N}, \sigma_1, \dots, \sigma_n \in \mathcal{S}, z_1, \dots, z_n \in X \text{ and } \varepsilon > 0 \right\}$$

is a neighborhood basis at x (for $\tau(\mathcal{S})$). Thus, to show the claim, it suffices to show that if $V \in \mathcal{V}_x$ (resp. $B \in \mathcal{B}_x$), then there exists $B \in \mathcal{B}_x$ (resp. $V \in \mathcal{V}_x$) such that $B \subseteq V$ (resp. $V \subseteq B$).

Consider first $V = \bigcap_{k=1}^n V_{(\sigma_k, z_k), \varepsilon}(x) \in \mathcal{V}_x$. Set $B := \bigcap_{k=1}^n B_\varepsilon^{\sigma_k} \in \mathcal{B}_x$. Using part a) we get that $B \subseteq V$.

On the other hand, consider $B = \bigcap_{k=1}^n B_\varepsilon^{\sigma_k}(x) \in \mathcal{B}_x$. Set $V := \bigcap_{k=1}^n V_{(\sigma_k, x), \varepsilon}(x) \in \mathcal{V}_x$. Using part a) we get that $B = V$.

Hence the claim follows.

Exercise 10

Let (X, τ) be topological vector space (over $\mathbb{K}$) and let $\ell : X \rightarrow \mathbb{K}$ be a linear functional.

The following conditions are equivalent:

- a) ℓ is continuous on X .
- b) ℓ is continuous at some point of X .
- c) $\sup\{\operatorname{Re} \ell(u) \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.
- d) $\inf\{\operatorname{Re} \ell(u) \mid u \in U\} > -\infty$ for some nonempty open $U \subseteq X$.
- e) $\sup\{|\ell(u)| \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.

$a) \Rightarrow b)$: This implication is obvious.

$b) \Rightarrow c)$: Assume that ℓ is continuous at $x_0 \in X$. Then ℓ is continuous at $0 \in X$. Indeed, if $\{y_\alpha\}$ is a net in X converging to 0, then $(x_0 + y_\alpha) \rightarrow_\alpha x_0$, so

$$\ell(x_0) + \ell(y_\alpha) = \ell(x_0 + y_\alpha) \rightarrow_\alpha \ell(x_0).$$

Hence $\ell(y_\alpha) \rightarrow_\alpha 0 = \ell(0)$, and the claim follows.

This implies that $\operatorname{Re} \ell$ is continuous at 0. In particular, there exists an open $U \subseteq X$ such that $\operatorname{Re} \ell(U) \subseteq (-1, 1)$, and it clearly follows that $c)$ holds.

$c) \Rightarrow d)$: Assume that $c)$ holds. So there exists some $M \in \mathbb{R}$ and some nonempty open $U \subseteq X$ such that $\operatorname{Re} \ell(U) \subseteq (-\infty, M]$. Then we have $\operatorname{Re} \ell(-u) = -\operatorname{Re} \ell(u) \in [-M, \infty)$ for all $u \in U$. Thus, $V := -U$ is a nonempty open subset of X such that $\inf\{\operatorname{Re} \ell(v) \mid v \in V\} \geq -M > -\infty$, showing that $d)$ holds.

$d) \Rightarrow e)$: Assume that $d)$ holds. We may assume that $m := \inf\{\operatorname{Re} \ell(u) \mid u \in U\} > -\infty$ for some nonempty open neighborhood U of 0 (because if it happens that $0 \notin U$, then we just pick any $x_0 \in U$, and replace U by $U' = U - x_0$).

By continuity of multiplication by scalars, we may pick $\delta > 0$ and a nonempty open $V \subseteq X$ such that $\lambda v \in U$ for all $\lambda \in B_\delta(0)$ and all $v \in V$.

Let $v \in V$ and pick $r \in (0, \delta)$. Then for all $t \in \mathbb{R}$ we have $r e^{it} v \in U$. Thus we get

$$m \leq \operatorname{Re}(\ell(r e^{it} v)) = \operatorname{Re}(r e^{it} \ell(v))$$

for all $t \in \mathbb{R}$. Choosing t such that $e^{it} \ell(v) = -|\ell(v)|$ gives $m \leq -r |\ell(v)|$, so $|\ell(v)| \leq -m/r$. Since this holds for every $v \in V$, we get

$$\sup\{|\ell(v)| \mid v \in V\} \leq -\frac{m}{r} < \infty,$$

so $e)$ holds.

$e) \Rightarrow a)$: Assume that $e)$ holds. By translating U if necessary, we may assume that $M := \sup\{|\ell(u)| \mid u \in U\} < \infty$ for some nonempty open neighborhood U of 0.

Let $x \in X$. To show that ℓ is continuous at x , let $\{x_i\}$ be a net in X converging to x . Pick $0 < r < 1$ and an open neighborhood V of 0 such that $\lambda v \in U$ for all $\lambda \in B_r(0)$ and all $v \in V$. Let $\varepsilon > 0$. Set

$$W := \varepsilon \left(\frac{r}{M+r+1} \right)^2 V.$$

Since W is an open neighborhood of 0, we may find i_0 such that $(x_i - x) \in W$ whenever $i \gtrsim i_0$.

Now, as $\frac{r}{M+r+1} < r$, we have that $\frac{r}{M+r+1} V \subseteq U$, which gives that $W \subseteq \frac{\varepsilon r}{M+r+1} U$.

Hence, we get that

$$(x_i - x) \in \frac{\varepsilon r}{M+r+1} U \quad \text{whenever } i \gtrsim i_0.$$

Thus,

$$|\ell(x_i) - \ell(x)| = |\ell(x_i - x)| \leq \frac{\varepsilon r}{M+r+1} M < \varepsilon r < \varepsilon$$

whenever $i \gtrsim i_0$. This shows that $\ell(x_i) \rightarrow_i \ell(x)$. Thus we have shown that ℓ is continuous at x . It follows that $a)$ holds.

Exercise 11

Let X, X' be vector spaces (over $\mathbb{K}$). Let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote a family of seminorms on X (resp. X') and let τ (resp. τ') denote the topology on X (resp. X') induced by $\mathcal{S}$ (resp. $\mathcal{S}'$), that is, $\tau = \tau(\mathcal{S})$ (resp. $\tau' = \tau(\mathcal{S}')$).

Let $T : X \rightarrow X'$ be a linear map, and consider T as map from (X, τ) to (X', τ') . The following statements are equivalent:

- a) T is continuous.
- b) T is continuous at 0.
- c) For each $\sigma' \in \mathcal{S}'$ there exist a nonempty finite subset F of $\mathcal{S}$ and $M > 0$ such that

$$\sigma'(T(x)) \leq M \max_{\sigma \in F} \{\sigma(x)\} \quad \text{for all } x \in X.$$

The implication a) $\Rightarrow$ b) is trivial. To show that b) $\Rightarrow$ c), assume that b) holds and let $\sigma' \in \mathcal{S}'$. Set $U' := \sigma'^{-1}((-1, 1)) = \{x' \in X' : \sigma'(x') < 1\}$, which is a τ' -open neighborhood of 0 in X' . Since $T(0) = 0$ and T is continuous at 0, there exists a τ -open neighborhood U of 0 in X such that $T(U) \subseteq U'$.

Using Exercise 9, we can find a finite nonempty subset F of $\mathcal{S}$ and $\varepsilon > 0$ such that $B_\varepsilon^F(0) \subseteq U$, i.e., $\{y \in X : \sigma(y) < \varepsilon \text{ for all } \sigma \in F\} \subseteq U$.

Set $M := 2/\varepsilon > 0$. Let $x \in X$ and set $m(x) := \max\{\sigma(x) : \sigma \in F\}$. To show that c) holds, it suffices to check that $\sigma'(T(x)) \leq M m(x)$. There are two possibilities:

- $m(x) > 0$.

In this case, we have that $\frac{1}{M m(x)} x = \frac{\varepsilon}{2 m(x)} x \in B_\varepsilon^F(0) \subseteq U$. Hence we get that $T(\frac{1}{M m(x)} x) \in U'$, i.e., $\frac{1}{M m(x)} \sigma'(T(x)) < 1$, that is, $\sigma'(T(x)) \leq M m(x)$, as desired.

- $m(x) = 0$.

This means that $\sigma(x) = 0$ for every $\sigma \in F$, and we have to deduce that $\sigma'(T(x)) = 0$. Let $\lambda \in \mathbb{F}$. Then we have that $\sigma(\lambda x) = |\lambda| \sigma(x) = 0 < \varepsilon$ for all $\sigma \in F$. Thus $\lambda x \in B_\varepsilon^F(0) \subseteq U$, so we get that $T(\lambda x) \in U'$. Hence $|\lambda| \sigma'(T(x)) = \sigma'(T(\lambda x)) < 1$. Since this is true for every $\lambda \in \mathbb{F}$, we must have that $\sigma'(T(x)) = 0$, as desired.

We have thereby shown that b) $\Rightarrow$ c).

Finally, assume that c) holds. Let $x \in X$ and let $\{x_\alpha\}$ be a net in X converging to x . To show that a) holds, it suffices to show that $T(x_\alpha) \rightarrow_\alpha T(x)$, i.e., $\sigma'(T(x_\alpha - x)) \rightarrow_\alpha 0$ for every $\sigma' \in \mathcal{S}'$.

Let $\sigma' \in \mathcal{S}'$ and $\varepsilon > 0$. Using the assumption, we may pick a (nonempty) finite subset F of $\mathcal{S}$ and $M > 0$ such that $\sigma'(T(y)) \leq M \max_{\sigma \in F} \{\sigma(y)\}$ for all $y \in X$. In particular, we have that

$$\sigma'(T(x_\alpha - x)) \leq M \max_{\sigma \in F} \{\sigma(x_\alpha - x)\} \text{ for all } \alpha.$$

Choose now α_0 such that $\sigma(x_\alpha - x) < \varepsilon/M$ for all $\sigma \in F$ and all $\alpha \gtrsim \alpha_0$. Then we clearly get that $\sigma'(T(x_\alpha - x)) < \varepsilon$ for all $\alpha \gtrsim \alpha_0$, which shows that $\sigma'(T(x_\alpha - x)) \rightarrow_\alpha 0$, as desired. Thus we have shown that c) $\Rightarrow$ a).

Exercise 12

Let X be a normed space (over $\mathbb{K}$).

a) Assume X is finite-dimensional. *Then the weak topology on X agrees with the norm topology:*

Since the weak topology on X is always weaker than the norm-topology, we have to show that it is stronger, that is, every norm-open subset of X is weak-open.

We may assume that $X \neq \{0\}$. Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for X , and $\{e_1, \dots, e_n\}$ denote the standard basis for $\mathbb{K}^n$. For $j = 1, \dots, n$, define $\varphi_j \in X^*$ by

$$\varphi_j(x) := [x]_{\mathcal{B}} \cdot e_j \quad (= \text{the } j\text{-th coordinate of } x \text{ w.r.t. } \mathcal{B}).$$

(Note that φ_j is continuous since it is linear and X is finite-dimensional).

Let U be a nonempty norm-open subset of X , and consider $x \in U$. We can then find $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$, where $B_\varepsilon(x) := \{y \in X : \|y - x\| < \varepsilon\}$.

Set $M := \max\{\|b_j\| : j = 1, \dots, n\}$, $\delta := \varepsilon/(Mn)$ and

$$V := \bigcap_{j=1}^n B_\delta^{|\varphi_j|}(x) = \{y \in X : |\varphi_j(y - x)| < \delta, j = 1, \dots, n\}.$$

Then V is a weak-open subset of X containing x . Moreover, $V \subseteq B_\varepsilon(x)$:

Indeed, let $y \in V$ and set $\lambda_j := [y - x]_{\mathcal{B}} \cdot e_j = \varphi_j(y - x)$, $j = 1, \dots, n$. Then $|\lambda_j| < \delta$ for each j , so

$$\begin{aligned} \|y - x\| &= \|\lambda_1 b_1 + \dots + \lambda_n b_n\| \\ &\leq \sum_{j=1}^n |\lambda_j| \|b_j\| \leq M \sum_{j=1}^n |\lambda_j| \\ &< Mn \delta = \varepsilon, \end{aligned}$$

i.e., $y \in B_\varepsilon(x)$.

Thus we get

$$x \in V \subseteq B_\varepsilon(x) \subseteq U.$$

This shows that every $x \in U$ is a weak-interior point of U , hence that U is weak-open.

b) Assume X is infinite-dimensional and let U be a weakly open neighbourhood of 0 in X (i.e., $0 \in U \subseteq X$ and $U \in \tau_{\text{weak}}$). *Then U contains an infinite-dimensional subspace of X :*

If $U = X$, then the assertion is trivial. So we may assume that $U \neq X$. Exercise 6 gives that there exist $\varepsilon > 0$ and $\varphi_1, \dots, \varphi_n \in X^*$ such that

$$\bigcap_{j=1}^n \{x \in X : |\varphi_j(x)| < \varepsilon\} \subseteq U.$$

In particular, we have $\bigcap_{j=1}^n \ker(\varphi_j) \subseteq U \neq X$. Now, consider the linear map $L : X \rightarrow \mathbb{F}^n$ defined by

$$L(x) = (\varphi_1(x), \dots, \varphi_n(x))$$

for all $x \in X$. Then $\ker(L) = \bigcap_{j=1}^n \ker(\varphi_j) \subseteq U$, and $L(X) \neq \{0\}$ (otherwise, all the φ_j 's would be equal to the zero functional on X , so we would have $\ker(L) = X \subseteq U \neq X$, a contradiction).

We claim that the subspace $\ker(L)$ is infinite-dimensional:

Assume (for contradiction) that $\ker(L)$ is finite-dimensional. Since X is infinite-dimensional, we must have that $\ker(L) \neq \{0\}$. Pick a basis $v_1, \dots, v_m$ for $\ker L$ and a basis $r_1, \dots, r_p$ for $L(X)$ (so $1 \leq p \leq n$). Next, pick $u_1, \dots, u_p \in X$ such that $L(u_k) = r_k$ for $k = 1, \dots, p$. Then one easily checks that X is spanned by the vectors $u_1, \dots, u_p, v_1, \dots, v_m$. Hence X is finite-dimensional, which gives a contradiction. Thus, $\ker(L)$ must be infinite-dimensional.

As $\ker(L)$ is contained in U , the desired assertion is proven.

c) Assume X is infinite-dimensional. Then the weak topology on X does not coincide with the norm topology:

Assume (for contradiction) that these two topologies agree. Let U denote the open unit ball in X , i.e. $U = \{x \in X : \|x\| < 1\}$. Then the assumption gives that U is weakly open, so U contains an infinite-dimensional subspace of X , say M . Pick $v \in M$, $v \neq 0$. Then $\lambda v \in M$ for every $\lambda \in \mathbb{F}$, i.e., $|\lambda| \|v\| = \|\lambda v\| < 1$ for every $\lambda \in \mathbb{F}$. This implies that $|\lambda| < 1/\|v\|$ for every $\lambda \in \mathbb{F}$, which is impossible.

Exercise 13

Let X be a normed space, $X \neq \{0\}$.

a) Assume that X^* is finite-dimensional. Then X is finite-dimensional too:

Indeed, we then have that $X^{**} = (X^*)^*$ is finite-dimensional. Let j denote the canonical map from X into X^{**} (given by $x \mapsto j_x$, where $j_x(\varphi) = \varphi(x)$). Since j is a linear isometry, and therefore injective, we get that $\dim(X) = \dim(j(X)) \leq \dim(X^{**}) < \infty$.

b) X is finite-dimensional if and only if the weak*-topology on X^* agrees with the norm-topology on X^* :

Assume that X is finite-dimensional. Let j be as above. Then

$$\dim(j(X)) = \dim(X) = \dim(X^*) = \dim(X^{**}),$$

so $j(X) = X^{**}$. So the weak*-topology on X^* , being the weak topology on X^* determined by $j(X) = (X^*)^*$, is the same as the weak topology on the normed space X^* . Since X^* is finite-dimensional, we get that this topology agrees with the norm-topology on X^* .

Conversely, assume that the weak*-topology on X^* agrees with the norm-topology. Now, the weak topology on X^* is weaker than the norm topology, and stronger than the weak*-topology. So all these three topologies on X^* coincide, in particular, the weak topology on X^* agrees with the norm topology. This forces X^* to be finite-dimensional, which in turn forces X to be finite-dimensional by a).

c) X is finite-dimensional if and only if the closed unit ball B^* in X^* is compact (w.r.t. the norm-topology on X^*):

Using a) and results shown earlier, we get that

$$X \text{ is finite-dimensional} \Leftrightarrow X^* \text{ is finite-dimensional} \Leftrightarrow B^* \text{ is norm-compact in } X^*.$$

d) Assume that X is infinite-dimensional. Let W be a weak*-open neighborhood of 0 in X^* . Then W contains an infinite-dimensional subspace of X^* :

The proof is similar to the proof of Exercise 12 c), so we just sketch the proof. We may choose $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that

$$\bigcap_{j=1}^n \{\varphi \in X^* : |\varphi(x_j)| < \varepsilon\} \subseteq W.$$

Consider now the linear map $L' : X^* \rightarrow \mathbb{F}^n$ given by $L'(\varphi) = (\varphi(x_1), \dots, \varphi(x_n))$ for all $\varphi \in X^*$. Clearly, we get that $\ker(L') \subseteq W$. Arguing essentially as in the proof of Exercise 12 c), one gets that $\ker(L')$ is infinite-dimensional, which shows the assertion.