

MAT4450 - Spring 2026 - Solutions of exercises - Set 4

Exercise 14

Let X be a vector space (over $\mathbb{K}$) and let A be a nonempty subset of X . We recall that $a \in A$ is called an internal point of A if for all $x \in X \setminus \{0\}$ there exists some $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in \mathbb{K}$ satisfying $|\lambda| < \varepsilon$. We let A^{int} denote the set of all internal points of A .

a) Assume that X is a topological vector space and let A° denotes the interior of A .

Then we have $A^\circ \subseteq A^{\text{int}}$:

Let $a \in A^\circ$ and $x \in X \setminus \{0\}$. We may then find a neighborhood U of a such that $U \subseteq A$. Next, by continuity of the map $\lambda \mapsto a + \lambda x$, we may find $\varepsilon > 0$ such that $a + \lambda x \in U \subseteq A$ whenever $\lambda \in \mathbb{K}$, $|\lambda| < \varepsilon$. Hence, $a \in A^{\text{int}}$.

b) Consider $X = \mathbb{R}^2$ with its usual topology and set

$$A = \{(x, y) \in [-1, 1] \times [-1, 1] : x^2 \leq y \text{ or } y \leq 0\} \subseteq \mathbb{R}^2.$$

Then A is not convex and $A^\circ \neq A^{\text{int}}$:

Indeed, making a drawing of A , one easily sees that A is not convex and $0 \in A^{\text{int}} \setminus A^\circ$.

c) Assume that X is a finite-dimensional normed space and A is convex.

Then we have $A^\circ = A^{\text{int}}$:

We sketch a proof. Let $a \in A^{\text{int}}$. By replacing A with $A - a$ if necessary, we may assume that $a = 0$. We may then find a basis $\{x_1, \dots, x_n\}$ for X such that $\lambda x_j \in A$ for every $\lambda \in T := \{\lambda \in \mathbb{K} : |\lambda| = 1\}$ and every $j \in \{1, \dots, n\}$. Let $\|\cdot\|'$ be the norm on X given by

$$\left\| \sum_{j=1}^n c_j x_j \right\|' := \sum_{j=1}^n |c_j|$$

whenever $c_1, \dots, c_n \in \mathbb{K}$. Now, if $x = \sum_{j=1}^n c_j x_j \in X$ is such that $\|x\|' < 1$, and we choose $\lambda_j \in T$ such that $c_j = \lambda_j |c_j|$ for every j , then x can be written as a convex combination of the vectors $0, \lambda_1 x_1, \dots, \lambda_n x_n$. As these vectors all belong to A , and A is convex, this implies that $\{x \in X : \|x\|' < 1\} \subseteq A$. But all norms on X are equivalent, so this shows that $a = 0$ is an interior point of A , as desired.

d) Consider $X = C([0, 1], \mathbb{R})$ as a normed space (over $\mathbb{R}$) w.r.t. the norm $\|f\|_1 = \int_0^1 |f(t)| dt$, and $A = \{f \in X : \sup_{t \in [0, 1]} |f(t)| < 1\}$.

Then A is convex and $A^\circ \neq A^{\text{int}}$:

Since A is the open unit ball in X w.r.t. to the sup-norm, A is convex. Moreover, 0 is an interior point of A w.r.t. to the sup-norm, so it is an internal point of A (by a)).

However, 0 is not an interior point of A w.r.t. to $\|\cdot\|_1$. Indeed, let $\varepsilon > 0$ and consider $U_\varepsilon = \{f \in X : \|f\|_1 < \varepsilon\}$.

Assume first that $\varepsilon < 1$. Let then $g_\varepsilon \in X$ be the function given by $g_\varepsilon(t) = 1 - \varepsilon^{-1} t$ when $t \in [0, \varepsilon]$, and by $g_\varepsilon(t) = 0$ when $t \in [\varepsilon, 1]$. Then $\|g_\varepsilon\|_1 = \varepsilon/2 < 1$, so $g_\varepsilon \in U_\varepsilon$. But $\sup_{t \in [0, 1]} |g_\varepsilon(t)| = 1$, so $g_\varepsilon \notin A$. Hence, $U_\varepsilon \not\subseteq A$.

Next, if $\varepsilon \geq 1$, then $g_{1/2} \in U_{1/2} \subseteq U_\varepsilon$, while $g_{1/2} \notin A$, so $U_\varepsilon \not\subseteq A$ in this case too.

This implies that no $\|\cdot\|_1$ -neighborhood of 0 is contained in A , i.e., $0 \notin A^\circ$. Hence, $0 \in A^{\text{int}} \setminus A^\circ$, so $A^\circ \neq A^{\text{int}}$.

Exercise 15

Let X be a nonzero vector space over $\mathbb{R}$. A subset H of X is called an *hyperplane* in X if there exists a nonzero linear functional $\ell : X \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that $H = \{x \in X \mid \ell(x) = c\}$, in which case the sets $\{x \in X \mid \ell(x) \geq c\}$ and $\{x \in X \mid \ell(x) \leq c\}$ are called the *half-spaces* determined by H .

Let $A, B \subseteq X$ (both nonempty). We say that A and B can be *separated* if there exists a nonzero linear functional $\ell : X \rightarrow \mathbb{R}$ such that $\ell(x) \leq \ell(y)$ for all $x \in A$ and $y \in B$.

a) We have that A and B can be separated if and only if B and A can be separated:

Indeed, if A and B can be separated using ℓ , then B and A can be separated using $-\ell$. The converse is then clear by symmetry.

Moreover, A and B can be separated if and only if there exists a hyperplane H in X such that A is contained in one of the half-spaces of H and B is contained in the other:

Assume A and B can be separated using ℓ . Set $s := \sup\{\ell(x) : x \in A\}$ and $t := \inf\{\ell(y) : y \in B\}$. Then $s, t \in \mathbb{R}$ and $s \leq t$. Pick $a \in A$ and $b \in B$, and set $u = \ell(a), v = \ell(b)$. Then $u \leq s \leq t \leq v$. By linearity of ℓ , ℓ will take all the values in the interval $[u, v]$. Now, if $s = t$, set $c := s = t$; otherwise, pick $c \in [s, t]$. Then $c \in [u, v]$, so $H := \{x \in X : \ell(x) = c\}$ is a (nonempty) hyperplane in X , and it is obvious that $A \subseteq \{x \in X \mid \ell(x) \leq c\}$ and $B \subseteq \{x \in X \mid \ell(x) \geq c\}$. Thus the RHS holds. The proof of the reverse implication is straightforward.

b) The real version of the algebraic Hahn-Banach separation theorem can be formulated as follows:

Let A, B be disjoint nonempty convex subsets of X . If A or B contains an internal point, then A and B can be separated.

This follows readily from the definition of what it means that A and B can be separated.

c) Consider $X := \{f : \mathbb{N} \rightarrow \mathbb{R} \mid f(j) = 0 \text{ for all } j\text{'s, except finitely many}\}$ as a vector space over $\mathbb{R}$ w.r.t. pointwise addition and scalar multiplication. For $f \in X \setminus \{0\}$, set

$$k_f := \min\{k \in \mathbb{N} : f(j) = 0 \text{ for all } j \geq k + 1\}.$$

Then define

$$A := \{f \in X \setminus \{0\} : f(k_f) > 0\} \quad \text{and} \quad B := \{f \in X \setminus \{0\} : f(k_f) < 0\}.$$

Then A and B are nonempty convex subsets of X , none of which contains an internal point:

For $k \in \mathbb{N}$ we let $e_k \in X$ be defined by $e_k(j) = 1$ if $j = k$ and $e_k(j) = 0$ if $j \neq k$.

It is clear that $e_1 \in A$ and $-e_1 \in B$, so A and B are nonempty. Assume $f, g \in A$ and $t \in (0, 1)$. We may assume $0 < k_f \leq k_g$. Set $h := (1 - t)f + tg$. Then $k_h = k_g$ and

$$h(k_h) = \begin{cases} t \cdot g(k_g) > 0 & \text{if } k_f < k_g, \\ (1 - t) \cdot f(k_f) + t \cdot g(k_g) > 0 & \text{if } k_f = k_g. \end{cases}$$

Thus $h \in A$. This shows that A is convex. The proof that B is convex is similar.

Next, assume (for contradiction) that A contains an internal point, say f . Choose $k > k_f$ and set $g := -e_k$. Let now $\varepsilon > 0$ be such that $f + \lambda g \in A$ for all $\lambda \in \mathbb{K}, |\lambda| < \varepsilon$. Then we have $h := f + \varepsilon/2 \cdot g \in A$. But $k_h = k$ and $h(k_h) = h(k) = -\varepsilon/2 < 0$. Thus, $h \notin A$, which gives a contradiction. So A does not contain any internal point. Similarly, one shows that B has no internal point either.

Moreover, A and B can not be separated:

Let $\ell : X \rightarrow \mathbb{R}$ be linear and assume $\ell(f) \leq \ell(g)$ for all $f \in A$ and $g \in B$. We have to show that $\ell = 0$.

Set $s := \sup\{\ell(f) : f \in A\}$ and $t := \inf\{\ell(g) : g \in B\}$, so $s \leq t$. We claim that $s \leq 0$.

Indeed, assume $s > 0$. Then $t > 0$. But for $g \in B$ and $n \in \mathbb{N}$, we have

$$\ell\left(\frac{1}{n} \cdot g\right) = \frac{1}{n} \cdot \ell(g) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\frac{1}{n} \cdot g \in B$, this implies that $t \leq 0$. Thus we have obtained a contradiction, so we must have $s \leq 0$. (A similar argument can be used to show that $0 \leq t$, but we won't need this.)

Since $s \leq 0$, we have $\ell(f) \leq 0$ for all $f \in A$. Let now $h \in X, h \neq 0$. Pick then $k \in \mathbb{N}$ such that $k > k_h$. Then $h + \frac{1}{n} \cdot e_k \in A$, so

$$0 \geq \ell\left(h + \frac{1}{n} \cdot e_k\right) = \ell(h) + \frac{1}{n} \cdot \ell(e_k) \rightarrow \ell(h) \quad \text{as } n \rightarrow \infty.$$

Thus $\ell(h) \leq 0$.

This shows that $\ell(h) \leq 0$ for every $h \in X \setminus \{0\}$. Hence for any $h \in X \setminus \{0\}$, we also have $-\ell(h) = \ell(-h) \leq 0$, that is, $\ell(h) \geq 0$. It follows that $\ell(h) = 0$ for every $h \in X \setminus \{0\}$, i.e., $\ell = 0$.
bigskip

Exercise 16

Here is an example of two nonempty disjoint (unbounded) closed convex subsets F and K of $\mathbb{R}^2$ where the conclusion of the Hahn-Banach separation theorem nr. III does not hold:

Set $F := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$ and $K := \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$. Then there is no $\varphi \in (\mathbb{R}^2)^*$ such that

$$\sup\{\varphi(x, y) : (x, y) \in K\} < \inf\{\varphi(x, y) : (x, y) \in F\}.$$

This is easy to see by making a drawing, or by writing $(\mathbb{R}^2)^* = \{\varphi_{(a,b)} : (a, b) \in \mathbb{R}^2\}$, where $\varphi_{(a,b)}(x, y) := ax + by$, and studying the possible values of both sides of this inequality.

We note that the algebraic Hahn-Banach separation theorem guarantees that there exists a nonzero $\varphi \in (\mathbb{R}^2)^*$ such that $\sup\{\varphi(x, y) : (x, y) \in K\} \leq \inf\{\varphi(x, y) : (x, y) \in F\}$. This happens exactly when $\varphi = \varphi_{(0,b)}$, $b > 0$; both sides of the inequality are then equal to 0.

Exercise 17 (= Exercise 2.4.6 in Pedersen's book)

Let $\{x_n\}$ be a sequence in a normed space X , such that $\varphi(x_n) \rightarrow \varphi(x)$ for some $x \in X$ and all $\varphi \in X^*$. (This says that $x_n \rightarrow x$ in the weak topology of X). Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Then there is a convex combination y of vectors in $X_m := \{x_n : n \geq m\}$ such that $\|x - y\| < \varepsilon$:

We first recall that if A is a nonempty subset of X , then $\text{co}(A)$ denotes the convex subset of X consisting of all convex combinations of vectors in A .

Set $S_m := \text{co}(X_m)$. Since S_m is a convex subset of X , we know that $\overline{S_m}^{\|\cdot\|} = \overline{S_m}^{\text{weak}}$. Since $\{x_n\}_{n \geq m}$ is a sequence in S_m converging weakly to x , this gives that $x \in \overline{S_m}^{\|\cdot\|}$. Thus, given $\varepsilon > 0$, there exists some $y \in S_m$ such that $\|x - y\| < \varepsilon$, as was to be shown.