

MAT4450 - Spring 2026 - Solutions of exercises - Set 5

Exercise 18 Let X, Y, Z be normed spaces (over $\mathbb{K}$).

a) The map $T \mapsto T^\sharp$ from $\mathcal{B}(X, Y)$ into $\mathcal{B}(Y^*, X^*)$ is linear:

Let $S, T \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{K}$. Then for all $\psi \in Y^*$ we have

$$(\lambda S + T)^\sharp(\psi) = \psi \circ (\lambda S + T) = \lambda \psi \circ S + \psi \circ T = (\lambda S^\sharp + T^\sharp)(\psi).$$

Thus, $(\lambda S + T)^\sharp = \lambda S^\sharp + T^\sharp$.

b) Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. Then $(S \circ T)^\sharp = T^\sharp \circ S^\sharp$:

For all $\psi \in Z^*$ we have

$$(S \circ T)^\sharp(\psi) = \psi \circ (S \circ T) = (\psi \circ S) \circ T = (S^\sharp(\psi)) \circ T = T^\sharp(S^\sharp(\psi)) = (T^\sharp \circ S^\sharp)(\psi).$$

Thus, $(S \circ T)^\sharp = T^\sharp \circ S^\sharp$.

Exercise 19

a) Let $n, m \in \mathbb{N}$, and choose some norms on $\mathbb{K}^n$ and $\mathbb{K}^m$. Let A be a $m \times n$ matrix over $\mathbb{K}$, and $T : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be the linear map having A as its standard matrix. We identify $\mathbb{K}^n$ with $(\mathbb{K}^n)^*$ via the map $y \mapsto \varphi_y$, where $\varphi_y(x) := x \cdot y = \sum_{i=1}^n x_i y_i$. Similarly, we identify $\mathbb{K}^m$ with $(\mathbb{K}^m)^*$.

The standard matrix of the adjoint operator $T^\sharp : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is the transpose of A :

Let $y \in \mathbb{K}^m$. Then, for all $x \in \mathbb{K}^n$, we have

$$[T^\sharp(y)](x) = (y \circ T)(x) = y(T(x)) = (Ax) \cdot y = x^t A^t y = x \cdot (A^t y) = [A^t y](x).$$

Thus $T^\sharp(y) = A^t y$.

b) Let X be a normed space over $\mathbb{K}$, and let $\varphi \in X^*$, i.e., $\varphi \in \mathcal{B}(X, \mathbb{K})$. Identify $\mathbb{K}$ with $(\mathbb{K})^*$ in the obvious way (that is, we consider $\lambda \in \mathbb{K}$ as the linear functional on $\mathbb{K}$ given by $\lambda(\mu) = \lambda\mu$ for all $\mu \in \mathbb{K}$).

Then the adjoint map $\varphi^\sharp \in \mathcal{B}(\mathbb{K}, X^*)$ is given by $\varphi^\sharp(\lambda) = \lambda\varphi$ for all $\lambda \in \mathbb{K}$:

Let $\lambda \in \mathbb{K}$. Then, for all $x \in X$, we have

$$[\varphi^\sharp(\lambda)](x) = [\lambda \circ \varphi](x) = \lambda(\varphi(x)) = \lambda\varphi(x) = [\lambda\varphi](x),$$

which proves the assertion.

Exercise 20

Consider $X = \ell^1(\mathbb{N}, \mathbb{K})$ as a normed space w.r.t. the $\|\cdot\|_1$ -norm. Recall that $\ell^\infty(\mathbb{N}, \mathbb{K})$ (with the $\|\cdot\|_\infty$ -norm) may be identified with X^* via the isometric isomorphism $g \mapsto \varphi_g$, where $\varphi_g(f) := \sum_{n=1}^\infty f(n)g(n)$ for all $f \in \ell^1(\mathbb{N}, \mathbb{K})$ whenever $g \in \ell^\infty(\mathbb{N}, \mathbb{K})$.

Set $Y := c_0(\mathbb{N}, \mathbb{K}) = \{g \in \ell^\infty(\mathbb{N}, \mathbb{K}) \mid \lim_{n \rightarrow \infty} g(n) = 0\}$. Recall also that X may be identified with Y^* (when Y is equipped with the $\|\cdot\|_\infty$ -norm), via the isometric isomorphism $f \mapsto \psi_f$, where $\psi_f(g) = \sum_{n=1}^\infty f(n)g(n)$ for all $f \in X = \ell^1(\mathbb{N}, \mathbb{K})$ whenever $g \in Y = c_0(\mathbb{N}, \mathbb{K})$.

a) Let $T : X \rightarrow Y$ be the linear map given by $[T(f)](n) = \sum_{m=n}^\infty f(m)$ for all $f \in X$ and $n \in \mathbb{N}$.

Then T is bounded:

Indeed, let $f \in X$. Then we have that

$$|[T(f)](n)| \leq \sum_{m=n}^{\infty} |f(m)| \leq \sum_{m=1}^{\infty} |f(m)| = \|f\|_1$$

for all $n \in \mathbb{N}$. So $\|T(f)\|_{\infty} \leq \|f\|_1$. Hence, T is bounded, with $\|T\| \leq 1$.

An expression for $T^{\sharp} \in \mathcal{B}(Y^*, X^*) = \mathcal{B}(X, X^*)$ is as follows.

Let $f \in Y^* = X$. Then, for all $h \in X$, we get (using Fubini)

$$\begin{aligned} [T^{\sharp}(f)](h) &= (f \circ T)(h) = f(T(h)) = \sum_{n=1}^{\infty} f(n)[T(h)](n) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} f(n)h(m) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^m h(m)f(n) = \sum_{m=1}^{\infty} h(m)g(m) = [g](h), \end{aligned}$$

where $g \in X^* = \ell^{\infty}(\mathbb{N}, \mathbb{K})$ is given by $g(m) := \sum_{n=1}^m f(n)$ for all $m \in \mathbb{N}$.

Thus, $[T^{\sharp}(f)](m) = \sum_{n=1}^m f(n)$ for all $m \in \mathbb{N}$.

b) Consider Y as a subspace of X^* . Then Y is norm-closed in X^* :

Let $\{g_m\}$ be a sequence in Y such that $\|g_m - g\|_{\infty} \rightarrow 0$ as $m \rightarrow \infty$ for some $g \in X^* = \ell^{\infty}(\mathbb{N}, \mathbb{K})$. Then for all $n, m \in \mathbb{N}$ we have

$$|g(n)| \leq |g(n) - g_m(n)| + |g_m(n)| \leq \|g - g_m\|_{\infty} + |g_m(n)|$$

Let $\varepsilon > 0$. Then choose first $m \in \mathbb{N}$ such that $\|g - g_m\|_{\infty} < \varepsilon/2$. Then choose $N \in \mathbb{N}$ such that $|g_m(n)| < \varepsilon/2$ for all $n \geq N$. Then we get that $|g(n)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n \geq N$. This shows that $g(n) \rightarrow 0$ as $n \rightarrow \infty$, i.e., $g \in Y$, as desired.

Moreover, we have that $(Y_{\perp})^{\perp} = X^*$ (so $Y \neq (Y_{\perp})^{\perp}$):

We first show that $Y_{\perp} = \{0\}$: Let $f \in Y_{\perp}$. So $f \in X$ and $\sum_{n=1}^{\infty} f(n)g(n) = 0$ for all $g \in Y$. Choosing g to be the indicator function of the set $\{m\}$ for any $m \in \mathbb{N}$ gives that $f(m) = 0$. Thus $f = 0$, as desired. Using this we get $(Y_{\perp})^{\perp} = \{0\}^{\perp} = X^*$.

Exercise 21 [= Exercise 2.3.8 in Pedersen's book]

Let X and Y be Banach spaces (over $\mathbb{K}$), and assume $T \in \mathcal{B}(X, Y)$ is such that $T(X)$ is closed in Y . Let $T^{\sharp} : Y^* \rightarrow X^*$ denote the adjoint of T .

We have $T^{\sharp}(Y^*) = (\ker(T))^{\perp}$:

Let $\psi \in Y^*$. Then, for all $x \in \ker(T)$, we have

$$[T^*(\psi)](x) = (\psi \circ T)(x) = \psi(T(x)) = \psi(0) = 0.$$

Thus, $T^*(\psi) \in \ker(T)^{\perp}$. This shows that $T^*(Y^*) \subseteq \ker(T)^{\perp}$. To show the reverse inclusion, follow the sketch of the proof given as a hint in Pedersen's book.

Exercise 22

Let X be a normed space (over $\mathbb{K}$) and N be a (norm-)closed subspace of X^* .

Then N is separating for X if and only if N is weak*-dense in X^* :

Recall that $N_\perp = \{x \in X : \psi(x) = 0 \text{ for all } \psi \in N\}$. One readily checks that N is separating for X if and only if $N_\perp = \{0\}$. Setting $Z := \overline{N}^{\text{weak}^*} \subseteq X^*$, it therefore suffices to show that $N_\perp = \{0\}$ if and only if $Z = X^*$.

Now, we have seen in a lecture that $Z = (N_\perp)^\perp$, and it is elementary to check that $Z_\perp = N_\perp$. Thus, if $N_\perp = \{0\}$, then $Z = \{0\}^\perp = X^*$. Conversely, if $Z = X^*$, then

$$N_\perp = Z_\perp = (X^*)_\perp = \{x \in X : \psi(x) = 0 \text{ for all } \psi \in X^*\} = \{0\}$$

(since X^* is separating for X).

Exercise 23

Let H be a Hilbert space and M be a closed subspace of H . Consider the annihilator of M

$$M^\perp = \{\varphi \in H^* : \varphi(x) = 0 \text{ for all } x \in M\}$$

and the orthogonal complement of M

$$M^{\perp_H} = \{y \in H : \langle x, y \rangle = 0 \text{ for all } x \in M\}.$$

a) Let $\Phi : H \rightarrow H^*$ denote the conjugate linear surjective isometric map given by $\Phi(y) = \varphi_y$ for each $y \in H$, where $\varphi_y(x) = \langle x, y \rangle$ for all $x \in H$.

We have $M^\perp = \Phi(M^{\perp_H})$:

Let $\varphi \in M^\perp$. Write $\varphi = \Phi(y) = \varphi_y$ with $y \in H$. Then, for every $x \in M$, we have

$$0 = \varphi(x) = \varphi_y(x) = \langle x, y \rangle.$$

Thus, $y \in M^{\perp_H}$, so $\varphi = \Phi(y) \in \Phi(M^{\perp_H})$.

Conversely, assume $\varphi \in \Phi(M^{\perp_H})$. So $\varphi = \Phi(y) = \varphi_y$ with $y \in M^{\perp_H}$. Then, for every $x \in M$, we have

$$\varphi(x) = \varphi_y(x) = \langle x, y \rangle = 0.$$

Thus, $\varphi \in M^\perp$.

b) Consider the quotient space H/M with the quotient norm.

Then H/M is isometrically isomorphic with $M^{\perp_H}$:

Let $Q : H \rightarrow H$ denote the orthogonal projection of H onto $M^{\perp_H}$. Then Q is bounded, has range $M^{\perp_H}$ and $\ker(Q) = M$. So the map $\tilde{Q} : H/M \rightarrow M^{\perp_H}$, given by $\tilde{Q}(\xi + M) = Q(\xi)$ for all $\xi \in H$, is an isomorphism such that $\|\tilde{Q}\| = \|Q\|$.

In fact, $\tilde{Q}$ is isometric: indeed, $P := I - Q$ is the orthogonal projection of H onto M , so for each $\xi \in H$ we have

$$\|\tilde{Q}(\xi + M)\| = \|Q(\xi)\| = \|(I - P)(\xi)\| = \|\xi - P(\xi)\| = \inf\{\|\xi - m\| : m \in M\} = \|\xi + M\|.$$

For completeness, we remark that H/M may be organized as a Hilbert space. Indeed, we can use $\tilde{Q}$ to transport the inner product of $M^{\perp_H}$ to H/M , by setting

$$\langle \xi + M, \eta + M \rangle := \langle \tilde{Q}(\xi + M), \tilde{Q}(\eta + M) \rangle = \langle Q(\xi), Q(\eta) \rangle$$

for all $\xi, \eta \in H$. It is then straightforward to check that H/M becomes a Hilbert space w.r.t. this inner product.

c) Let $T \in \mathcal{B}(H)$. Recall that $T^* \in \mathcal{B}(H)$ denotes the Hilbert space adjoint of T .

We have $T^\sharp = \Phi T^* \Phi^{-1}$.

Equivalently, we have to show that $T^* = \Phi^{-1} T^\sharp \Phi$.

Let $y \in H$. Then for all $x \in H$ we have

$$(\varphi_y \circ T)(x) = \varphi_y(T(x)) = \langle T(x), y \rangle = \langle x, T^*(y) \rangle = \varphi_{T^*(y)}(x).$$

Thus, $\varphi_y \circ T = \varphi_{T^*(y)}$. Hence,

$$\begin{aligned} (\Phi^{-1} T^\sharp \Phi)(y) &= \Phi^{-1}(T^\sharp(\Phi(y))) = \Phi^{-1}(T^\sharp(\varphi_y)) \\ &= \Phi^{-1}(\varphi_y \circ T) = \Phi^{-1}(\varphi_{T^*(y)}) \\ &= T^*(y). \end{aligned}$$

This shows the desired equality.

Exercise 24

Let X be a normed space (over $\mathbb{K}$) and M be a (norm-)closed subspace of X . Let $Q : X \rightarrow X/M$ denote the quotient map.

Then the adjoint operator $Q^* : (X/M)^* \rightarrow X^*$ is isometric:

Let $\psi \in (X/M)^*$. Since $\|Q^*\| = \|Q\| \leq 1$, we get that $\|Q^*(\psi)\| \leq \|Q^*\| \|\psi\| \leq \|\psi\|$.

To show the reverse inequality, we will use the fact that if X is a normed space and $\varphi \in X^*$, then

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| < 1\}.$$

(You should prove this fact if you have not seen it before.)

Let $y + M \in X/M$ satisfy that $\|y + M\| < 1$. By definition of the quotient norm, we can then pick $m \in M$ such that $\|y + m\| < 1$. Set $z := y + m \in X$. Then $\|z\| < 1$, and we get that

$$\begin{aligned} |\psi(y + M)| &= |\psi(z + M)| = |\psi(Q(z))| = |(Q^*(\psi))(z)| \\ &\leq \sup\{|[Q^*(\psi)](x)| : x \in X, \|x\| < 1\} \\ &= \|Q^*(\psi)\|. \end{aligned}$$

This gives that

$$\|\psi\| = \sup\{|\psi(y + M)| : y + M \in X/M, \|y + M\| < 1\} \leq \|Q^*(\psi)\|.$$

Altogether, we have shown that $\|Q^*(\psi)\| = \|\psi\|$, as desired.