

## MAT4450 - Spring 2026 - Solutions of exercises - Set 6

### Exercise 25

This exercise will be a part of the compulsory assignment this spring.

### Exercise 26

Let  $X$  be a vector space (over  $\mathbb{K}$ ).

a) Let  $A$  be a nonempty subset of  $X$  and set

$$\text{co}(A) := \left\{ \sum_{j=1}^n \lambda_j a_j \mid n \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in [0, 1], \sum_{j=1}^n \lambda_j = 1, a_1, \dots, a_n \in A \right\}.$$

Then  $\text{co}(A)$  is the least convex subset of  $X$  containing  $A$ :

We first remark that an easy induction argument shows that the following statement holds for every  $n \in \mathbb{N}$ : If  $C$  is a convex subset of  $X$  and  $u_1, \dots, u_n \in C$ , then any convex combination of the  $u_j$ 's belongs to  $C$ , that is, if  $\lambda_1, \dots, \lambda_n \in [0, 1]$ , and  $\sum_{j=1}^n \lambda_j = 1$ , then  $\sum_{j=1}^n \lambda_j u_j \in C$ .

Next, we note that  $A \subseteq \text{co}(A)$  (take  $n = 1$ ). Moreover,  $\text{co}(A)$  is convex:

Indeed, if  $x = \sum_{j=1}^n \lambda_j a_j$  and  $y = \sum_{k=1}^m \mu_k b_k$  both lies in  $\text{co}(A)$  (with obvious assumptions on the  $a_j$ 's, the  $b_k$ 's, the  $\lambda_j$ 's and the  $\mu_k$ 's), then for each  $0 \leq t \leq 1$ , we have

$$(1-t)x + ty = \sum_{j=1}^n (1-t)\lambda_j a_j + \sum_{k=1}^m t\mu_k b_k \in \text{co}(A)$$

since

$$\sum_{j=1}^n (1-t)\lambda_j + \sum_{k=1}^m t\mu_k = (1-t) \sum_{j=1}^n \lambda_j + t \sum_{k=1}^m \mu_k = 1-t+t=1.$$

Finally, assume that  $C$  is a convex subset of  $X$  containing  $A$ . Then  $\text{co}(A) \subseteq C$ :

Indeed, assume that  $x \in \text{co}(A)$ . Write  $x = \sum_{j=1}^n \lambda_j a_j$  as a convex combination of  $a_1, \dots, a_n \in A$ . Then  $a_1, \dots, a_n \in C$ , so our first remark implies that  $x \in C$ .

This shows that  $\text{co}(A)$  is the least convex subset of  $X$  containing  $A$ , as desired.

b) Let  $C \neq \emptyset$  be a convex subset of  $X$ . Assume  $F$  is a face of  $C$ , and  $K$  is a face of  $F$ .

Then  $K$  is a face of  $C$ :

Consider  $x, y \in C$  and  $0 < t < 1$  such that  $(1-t)x + ty \in K$ . Since  $K \subseteq F$ , we have  $(1-t)x + ty \in F$ . Since  $F$  is a face of  $C$ , this implies that  $x, y \in F$ . Since  $K$  is a face of  $F$ , we get that  $x, y \in K$ , as desired.

### Exercise 28

Let  $(X, \tau)$  be a locally convex Hausdorff topological vector space and  $K$  be a nonempty compact convex subset of  $X$ .

Let  $\varphi \in (X, \tau)^*$  and set

$$m := \inf\{\text{Re } \varphi(x) : x \in K\}, M := \sup\{\text{Re } \varphi(x) : x \in K\}, s := \sup\{|\varphi(x)| : x \in K\}.$$

There exist  $x, y, z \in \text{ex}(K)$  such that

$$\text{Re } \varphi(x) = m, \text{Re } \varphi(y) = M, |\varphi(z)| = s.$$

Set  $F := \{x \in K : \operatorname{Re} \varphi(x) = m\}$ . As we have seen in a lecture,  $F$  is a compact face of  $K$ . Thus  $F$  is a compact convex subset of  $X$ , so the Krein-Milman theorem gives that  $\operatorname{ex}(F) \neq \emptyset$ . Letting  $x \in \operatorname{ex}(F)$ , we get that  $x \in \operatorname{ex}(K)$  and  $\operatorname{Re} \varphi(x) = m$ .

Next, set  $\psi := -\varphi$ . Then  $\operatorname{Re} \psi = -\operatorname{Re} \varphi$  and

$$\inf \operatorname{Re} \psi(K) = -\sup \operatorname{Re} \varphi(K) = -M.$$

Using what we have shown above (with  $\psi$  instead of  $\varphi$ ), we get that there exists  $y \in \operatorname{ex}(K)$  such that  $\operatorname{Re} \psi(y) = -M$ , i.e.,  $\operatorname{Re} \varphi(y) = M$ .

Finally, to show the existence of  $z$ , we consider the function  $|\varphi|$  on  $X$  and assume that  $\mathbb{K} = \mathbb{C}$  (the case  $\mathbb{K} = \mathbb{R}$  is similar, but easier). Since  $K$  is compact, there exists some  $x_0 \in K$  such that

$$|\varphi(x_0)| = s.$$

Pick then  $\theta \in \mathbb{R}$  such that  $|\varphi(x_0)| = e^{i\theta} \varphi(x_0)$ , and define  $\varphi' \in (X, \tau)^*$  by  $\varphi' := e^{i\theta} \varphi$ .

Note that  $\varphi'(x_0) = |\varphi(x_0)| = s \in \mathbb{R}$ , so  $\operatorname{Re} \varphi'(x_0) = s$ . Moreover, for every  $v \in K$ , we have

$$\operatorname{Re} \varphi'(v) \leq |\varphi'(v)| = |\varphi(v)| \leq s.$$

Thus,  $\sup\{\varphi'(x) : x \in K\} = s$ . Using the second assertion we have shown (with  $\varphi'$  instead of  $\varphi$ ), we get that there exists  $z \in \operatorname{ex}(K)$  such that  $\operatorname{Re} \varphi'(z) = s$ . This gives that

$$s = \operatorname{Re} \varphi'(z) \leq |\varphi'(z)| = |\varphi(z)| \leq s,$$

hence that  $|\varphi(z)| = s$ , as desired.

### Exercise 28

Let  $H$  be a complex Hilbert space  $\neq \{0\}$ . Set

$$B := \{\xi \in H : \|\xi\| \leq 1\} \text{ and } \mathcal{B} := \{T \in \mathcal{B}(H) : \|T\| \leq 1\}.$$

a) We have  $\operatorname{ex}(B) = \{\eta \in B : \|\eta\| = 1\}$ :

We first show that  $\{\eta \in B : \|\eta\| = 1\} \subseteq \operatorname{ex}(B)$ .

Let  $\eta \in B$ ,  $\|\eta\| = 1$ . Assume  $\eta = (1-t)\xi + t\xi'$  for  $\xi, \xi' \in B$  and  $0 < t < 1$ . Then we must have  $\|\xi\| = \|\xi'\| = 1$  (otherwise, using the triangle inequality, we would get  $\|\eta\| < 1$ ).

Observe now that if  $\operatorname{Re} \langle \xi, \xi' \rangle < 1$ , then we get that

$$\begin{aligned} 1 = \|\eta\|^2 &= \|(1-t)\xi + t\xi'\|^2 = (1-t)^2 + 2t(1-t) \operatorname{Re} \langle \xi, \xi' \rangle + t^2 \\ &< (1-t)^2 + 2t(1-t) + t^2 = 1, \end{aligned}$$

a contradiction. So we must have  $\operatorname{Re} \langle \xi, \xi' \rangle \geq 1$ .

Using the Cauchy-Schwarz inequality, we then get that

$$1 \leq \operatorname{Re} \langle \xi, \xi' \rangle \leq |\langle \xi, \xi' \rangle| \leq \|\xi\| \|\xi'\| = 1,$$

thus,  $\operatorname{Re} \langle \xi, \xi' \rangle = 1$  (in fact,  $\langle \xi, \xi' \rangle = 1$ ). So

$$\|\xi - \xi'\|^2 = \|\xi\|^2 - 2 \operatorname{Re} \langle \xi, \xi' \rangle + \|\xi'\|^2 = 1 - 2 + 1 = 0,$$

hence  $\xi = \xi'$ . This shows that  $\eta \in \operatorname{ex}(B)$ . Hence the asserted inclusion holds.

Next, we show that  $\text{ex}(B) \subseteq \{\eta \in B; \|\eta\| = 1\}$  holds:

Assume  $\eta \in B$ ,  $\|\eta\| \neq 1$ , so  $\|\eta\| < 1$ . It suffices to show that  $\eta \notin \text{ex}(B)$ .

- If  $\eta = 0$ , then, since  $0 = \frac{1}{2}\xi + \frac{1}{2}(-\xi)$  and  $\xi \neq -\xi$  for any  $\xi \in B \setminus \{0\}$ , we get that  $\eta = 0 \notin \text{ex}(B)$ .
- Suppose now that  $\eta \neq 0$ . Set  $\xi := \frac{1}{\|\eta\|}\eta$  and  $t := \|\eta\|$ . Then  $0 \in B$ ,  $\xi \in B$ ,  $0 < t < 1$ , and

$$(1-t)0 + t\xi = \|\eta\| \frac{1}{\|\eta\|} \eta = \eta.$$

As  $\xi \neq 0$ , this implies that  $\eta \notin \text{ex}(B)$ .

b) Let  $T \in \mathcal{B}(H)$ . Assume that  $T$  or  $T^*$  is isometric.

Then  $T \in \text{ex}(\mathcal{B})$ :

Assume first that  $T$  is an isometry. Then  $T$  is norm-preserving, so  $T \in \mathcal{B}$ . Now, let  $R, S \in \mathcal{B}$  and  $0 < t < 1$  be such that

$$T = (1-t)R + tS.$$

We have to show that  $R = S$ . By linearity of  $R$  and  $S$ , it suffices to show that  $R(\eta) = S(\eta)$  for every  $\eta \in H$  such that  $\|\eta\| = 1$ . So let  $\eta \in H$ ,  $\|\eta\| = 1$ . Since  $T$  is norm-preserving, we have  $\|T(\eta)\| = \|\eta\| = 1$ . Using a), we get that  $T(\eta) \in \text{ex}(B)$ . Moreover, we have

$$T(\eta) = (1-t)R(\eta) + tS(\eta), \text{ and } R(\eta), S(\eta) \in B \quad (1)$$

because  $\|R(\eta)\| \leq \|R\| \|\eta\| \leq 1$  and  $\|S(\eta)\| \leq \|S\| \|\eta\| \leq 1$ .

Since  $T(\eta) \in \text{ex}(B)$ , we conclude from (1) that  $R(\eta) = S(\eta)$ , as desired. It follows  $T \in \text{ex}(\mathcal{B})$ .

Assume now that  $T^*$  is an isometry. (The operator  $T$  is then often called a *co-isometry*).

Again, let  $R, S \in \mathcal{B}$  and  $0 < t < 1$  be such that  $T = (1-t)R + tS$ . Then we get that

$$T^* = ((1-t)R + tS)^* = (1-t)R^* + tS^* \quad (2)$$

and  $\|R^*\| = \|R\| \leq 1$ ,  $\|S^*\| = \|S\| \leq 1$ , so  $R^*, S^* \in \mathcal{B}$ .

Since  $T^*$  is an isometry, we know from the first part that  $T^* \in \text{ext}(\mathcal{B})$ . Hence it follows from (2) that  $R^* = S^*$ . Thus we get that  $R = (R^*)^* = (S^*)^* = S$ . This shows that  $T \in \text{ex}(\mathcal{B})$ , as desired.

We note for completeness that it may be shown that the converse also holds, i.e., that if  $T \in \text{ex}(\mathcal{B})$ , then  $T$  or  $T^*$  is isometric. The proof requires some tools we have not covered in this course.

## Exercise 29

Consider the Banach space  $X = (L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_1)$  (over  $\mathcal{F}$ ), where  $\mathcal{B}_{\mathbb{R}}$  denotes the  $\sigma$ -algebra of all Borel subsets of  $\mathbb{R}$  and  $\mu$  denotes the Lebesgue measure on  $\mathcal{B}_{\mathbb{R}}$ .

Consider the closed ball  $B := \{f \in X : \|f\|_1 \leq 1\}$ , which is clearly convex.

a) *The convex ball  $B$  has no extreme points:*

We follow the given hint and assume (for contradiction) that there exists some  $f \in \text{ex}(B)$ .

We first show that  $\|f\|_1 = 1$ . Set  $t := \|f\|_1 \in [0, 1]$ .

- If  $t = 0$ , i.e.,  $f = 0$ , then we can pick  $g \in B \setminus \{0\}$ , in which case we also have  $-g \in B \setminus \{0\}$ , and write  $f = 0 = \frac{1}{2}g + \frac{1}{2}(-g)$ , which shows that  $f \notin \text{ex}(B)$ .
- If  $t \in (0, 1)$ , then  $f \neq 0$  and  $f = t(\frac{1}{t}f) + (1-t)0$ , where  $\frac{1}{t}f$  and  $0$  both lie in  $B$ , which shows that  $f \notin \text{ex}(B)$ .

Thus, in both cases, we get a contradiction. So we must have  $t = 1$ , as desired.

We now consider the continuous function  $F(t) := \int_{(-\infty, t]} |f| d\mu$ ,  $t \in \mathbb{R}$ . Since

$$\lim_{t \rightarrow \infty} F(t) = \|f\|_1 = 1 \quad \text{and} \quad \lim_{t \rightarrow -\infty} F(t) = 0,$$

there exists (at least one)  $t_0 \in \mathbb{R}$  such that  $F(t_0) = 1/2$ .

Set  $f_1 := 2f \mathbf{1}_{(-\infty, t_0]}$  and  $f_2 := 2f \mathbf{1}_{(t_0, \infty]}$ . Then  $f_1, f_2 \in B$ , since

$$\|f_1\|_1 = 2F(t_0) = 2 \cdot \frac{1}{2} = 1 \quad \text{and}$$

$$\|f_2\|_1 = 2 \int_{\mathbb{R}} |f| \mathbf{1}_{(t_0, \infty]} d\mu = 2 \int_{\mathbb{R}} |f| d\mu - 2F(t_0) = 2 - 1 = 1.$$

Moreover, we have

$$\frac{1}{2} f_1 + \frac{1}{2} f_2 = f \mathbf{1}_{(-\infty, t_0]} + f \mathbf{1}_{(t_0, \infty]} = f.$$

As  $f \in \text{ex}(B)$ , this implies that  $f = f_1 = f_2$ , which is possible only if  $f = 0$ . This contradicts that  $\|f\|_1 = 1$ . Hence, we must have  $\text{ex}(B) = \emptyset$ .

b) *There is no topology on  $X$  making it a locally convex Hausdorff topological vector space such that  $B$  is compact:*

Assume  $\tau$  was such a topology on  $X$ . The Krein-Milman theorem would then imply that  $B$  had some extreme points, which is not the case. Hence, there is no such topology on  $X$ .

*This implies that  $(L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_1)$  can not be isomorphic to the dual space of  $(L^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_\infty)$ :*

If  $(L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_1)$  was isomorphic to the dual space of  $(L^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu), \|\cdot\|_\infty)$ , then we could equip  $X = L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$  with the weak\*-topology and  $B$  would then be compact by Alaoglu's theorem; this would contradict the statement above.