

MAT4450 - Spring 2026 - Solutions of exercises - Set 8

Exercise 34

Let $\mathcal{A}$ be a non-unital Banach algebra over $\mathbb{K}$. Set $\tilde{\mathcal{A}} = \{(a, \alpha) \mid a \in \mathcal{A}, \alpha \in \mathbb{K}\}$ and define addition, multiplication by scalars and product by

$$\begin{aligned}(a, \alpha) + (b, \beta) &= (a + b, \alpha + \beta), \\ \lambda(a, \alpha) &= (\lambda a, \lambda \alpha), \\ (a, \alpha)(b, \beta) &= (ab + \alpha b + \beta a, \alpha \beta)\end{aligned}$$

for $(a, \alpha), (b, \beta) \in \tilde{\mathcal{A}}$ and $\lambda \in \mathbb{K}$. As you may check for yourself, $\tilde{\mathcal{A}}$ becomes a unital algebra with unit $e := (0, 1)$, and the map $(a, \alpha) \mapsto \alpha$ is an algebra-homomorphism from $\tilde{\mathcal{A}}$ into $\mathbb{K}$, with kernel equal to $\mathcal{A}_0 := \{(a, 0) \mid a \in \mathcal{A}\}$.

It is usual to identify $\mathcal{A}$ with the ideal $\mathcal{A}_0$ and to write (a, α) as $a + \alpha e$ (since $(a, \alpha) = (a, 0) + \alpha(0, 1)$).

Define $\|\cdot\|$ on $\tilde{\mathcal{A}}$ by

$$\|a + \alpha e\| := \|a\| + |\alpha|$$

for every $a + \alpha e \in \tilde{\mathcal{A}}$.

Then $\|\cdot\|$ is a norm on $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{A}}$ is a unital Banach algebra, containing $\mathcal{A}$ as a closed ideal:

Let $a + \alpha e, b + \beta e \in \tilde{\mathcal{A}}$ and $\lambda \in \mathbb{K}$. Then $\|a + \alpha e\| = \|a\| + |\alpha| \geq 0$,

$$\|\lambda(a + \alpha e)\| = \|\lambda a + \lambda \alpha e\| = \|\lambda a\| + |\lambda \alpha| = |\lambda|(\|a\| + |\alpha|) = |\lambda|\|a + \alpha e\|,$$

$$\|a + \alpha e\| = 0 \Leftrightarrow \|a\| = 0 \text{ and } |\alpha| = 0 \Leftrightarrow a = 0 \text{ and } \alpha = 0 \Leftrightarrow a + \alpha e = 0,$$

$$\begin{aligned}\|(a + \alpha e) + (b + \beta e)\| &= \|a + b + (\alpha + \beta)e\| = \|a + b\| + |\alpha + \beta| \\ &\leq \|a\| + \|b\| + |\alpha| + |\beta| = (\|a\| + |\alpha|) + (\|b\| + |\beta|) \\ &= \|a + \alpha e\| + \|b + \beta e\|,\end{aligned}$$

$$\begin{aligned}\|(a + \alpha e)(b + \beta e)\| &= \|ab + \alpha b + \beta a + \alpha \beta e\| \\ &= \|ab + \alpha b + \beta a\| + |\alpha \beta| \\ &\leq \|ab\| + \|\alpha b\| + \|\beta a\| + |\alpha \beta| \\ &\leq \|a\|\|b\| + |\alpha|\|b\| + |\beta|\|a\| + |\alpha \beta| \\ &= (\|a\| + |\alpha|)(\|b\| + |\beta|) \\ &= \|a + \alpha e\|\|b + \beta e\|.\end{aligned}$$

This shows that $\|\cdot\|$ is a norm on $\tilde{\mathcal{A}}$ which is submultiplicative, i.e., $\tilde{\mathcal{A}}$ is a normed algebra. Further, $(\mathcal{A}, \|\cdot\|)$ is complete. Indeed, let $\{a_n + \alpha_n e\}$ be a Cauchy sequence in $\tilde{\mathcal{A}}$. As

$$\|(a_n + \alpha_n e) - (a_m + \alpha_m e)\| = \|a_n - a_m\| + |\alpha_n - \alpha_m|$$

for all $n, m \in \mathbb{N}$, we see that $\{a_n\}$ and $\{\alpha_n\}$ are both Cauchy sequences, in $\mathcal{A}$ and in $\mathbb{K}$, respectively. So there exist $a \in \mathcal{A}$ and $\alpha \in \mathbb{K}$ such that $a_n \rightarrow a$ and $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$, and it follows then readily that $a_n + \alpha_n e \rightarrow a + \alpha e$ as $n \rightarrow \infty$. This shows that $\tilde{\mathcal{A}}$ is complete as a normed space, hence that $\tilde{\mathcal{A}}$ is a Banach algebra.

As the algebra-homomorphism $\pi : \tilde{\mathcal{A}} \rightarrow \mathbb{K}$ given by $\pi(a + \alpha e) = \alpha$ is clearly bounded and has $\mathcal{A}$ as its kernel, we get that $\mathcal{A}$ is closed ideal of $\tilde{\mathcal{A}}$.

The quotient $\tilde{\mathcal{A}}/\mathcal{A}$ is isometrically isomorphic to $\mathbb{K}$:

Since the range of π is $\mathbb{K}$, we get that the map $\tilde{\pi} : \tilde{\mathcal{A}}/\mathcal{A} \rightarrow \mathbb{K}$ given by

$$\tilde{\pi}((a + \alpha e) + \mathcal{A}) := \pi(a + \alpha e) = \alpha$$

is an algebra-isomorphism. For each $a + \alpha e \in \tilde{\mathcal{A}}$, we have that

$$\|(a + \alpha e) + \mathcal{A}\| = \inf\{\|a + \alpha e - b\| : b \in \mathcal{A}\} = \inf\{\|a - b\| + |\alpha| : b \in \mathcal{A}\} = |\alpha| = \|\tilde{\pi}((a + \alpha e) + \mathcal{A})\|,$$

so $\tilde{\pi}$ is isometric.

Exercise 35

Consider the commutative Banach algebra $\mathcal{A} = C(\Omega, \mathbb{K})$, where Ω is a compact Hausdorff space. Let $\omega_0 \in \Omega$ and define $\varphi : \mathcal{A} \rightarrow \mathbb{K}$ by $\varphi(f) = f(\omega_0)$ for $f \in \mathcal{A}$.

a) Then φ is a continuous algebra-homomorphism from $\mathcal{A}$ into $\mathbb{K}$ (considered as a Banach algebra) satisfying $\|\varphi\| = 1$:

It is clear that φ is linear. It is also multiplicative. Indeed, let $f, g \in \mathcal{A}$. Then

$$\varphi(fg) = (fg)(\omega_0) = f(\omega_0)g(\omega_0) = \varphi(f)\varphi(g).$$

Thus, φ is an algebra-homomorphism. Moreover, since $|\varphi(f)| = |f(\omega_0)| \leq \|f\|_\infty$ for all $f \in \mathcal{A}$, φ is bounded, hence continuous, with $\|\varphi\| \leq 1$. As $\|1_\Omega\|_\infty = 1$ and $|\varphi(1_\Omega)| = 1_\Omega(\omega_0) = 1$, we get that $\|\varphi\| = 1$.

b) Consider the closed ideal of $\mathcal{A}$ given by $\mathcal{J} = \ker(\varphi)$. Then the Banach algebra $\mathcal{A}/\mathcal{J}$ is isometrically isomorphic to $\mathbb{K}$:

Since the range of φ is $\mathbb{K}$, we get that the map $\tilde{\varphi} : \mathcal{A}/\mathcal{J} \rightarrow \mathbb{K}$ given by $\tilde{\varphi}(f + \mathcal{J}) = \varphi(f)$ for each $f \in \mathcal{A}$ is a bounded algebra-isomorphism (with $\|\tilde{\varphi}\| = \|\varphi\| = 1$).

Let $f \in \mathcal{A}$. To show that $\tilde{\varphi}$ is isometric, we have to show that $|\tilde{\varphi}(f + \mathcal{J})| = \|f + \mathcal{J}\|$. Since $|\tilde{\varphi}(f + \mathcal{J})| = |\varphi(f)| = |f(\omega_0)|$ and $\|f + \mathcal{J}\| = \inf\{\|f - g\|_\infty : g \in \mathcal{J}\}$, we have to show that

$$|f(\omega_0)| = \inf\{\|f - g\|_\infty : g \in \mathcal{J}\}.$$

Since $\mathcal{J} = \{g \in \mathcal{A} \mid g(\omega_0) = 0\}$, we have that

$$|f(\omega_0)| = |(f - g)(\omega_0)| \leq \|f - g\|_\infty \quad \text{for all } g \in \mathcal{J},$$

Hence, $|f(\omega_0)|$ is a lower bound of $\{\|f - g\|_\infty : g \in \mathcal{J}\}$.

Let now $\varepsilon > 0$. Since f is continuous at ω_0 , we can find an open neighborhood U of ω_0 such that $|f(\omega) - f(\omega_0)| < \varepsilon$ for all $\omega \in U$. This gives that

$$\left| |f(\omega)| - |f(\omega_0)| \right| \leq |f(\omega) - f(\omega_0)| < \varepsilon \quad \text{for all } \omega \in U,$$

hence that $|f(\omega)| < |f(\omega_0)| + \varepsilon$ for all $\omega \in U$.

Set $E := \Omega \setminus U$. Then E and $\{\omega_0\}$ are disjoint closed subsets of Ω , so, by Urysohn's lemma, we can find $h \in \mathcal{A}$ such that $0 \leq h \leq 1$, $h = 1$ on E , and $h(\omega_0) = 0$.

Set $k := fh$. Then $k \in \mathcal{J}$, $f - k = 0$ on E , and

$$|(f - k)(\omega)| = |f(\omega)|(1 - h(\omega)) \leq |f(\omega)| < |f(\omega_0)| + \varepsilon \quad \text{for all } \omega \in U.$$

Thus we get that

$$\|f - k\|_\infty \leq |f(\omega_0)| + \varepsilon.$$

Since $k \in \mathcal{J}$, it follows that $|f(\omega_0)|$ is the greatest lower bound of $\{\|f - g\|_\infty : g \in \mathcal{J}\}$, as desired.

From now on, a unital Banach algebra $\mathcal{A}$ will always mean a complex Banach algebra having a unit e which satisfies that $\|e\| = 1$.

Exercise 36

Let $\mathcal{A}$ be a unital Banach algebra.

a) Let $a \in \text{GL}(\mathcal{A})$ and note that $0 \notin \text{sp}_{\mathcal{A}}(a)$. We have

$$\text{sp}_{\mathcal{A}}(a^{-1}) = \{ \lambda^{-1} \mid \lambda \in \text{sp}_{\mathcal{A}}(a) \}$$

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Since $\lambda^{-1}I - a^{-1} = -\lambda^{-1}a^{-1}(\lambda I - a)$, and $\lambda^{-1}a^{-1} \in \text{GL}(\mathcal{A})$, we get that

$$\lambda \in \text{sp}_{\mathcal{A}}(a) \Leftrightarrow \lambda^{-1} \in \text{sp}_{\mathcal{A}}(a^{-1})$$

This implies that

$$\{ \lambda^{-1} \mid \lambda \in \text{sp}_{\mathcal{A}}(a) \} \subseteq \text{sp}_{\mathcal{A}}(a^{-1}).$$

On the other hand, let $\mu \in \text{sp}_{\mathcal{A}}(a^{-1})$. Then $\mu \neq 0$, and with $\lambda := \mu^{-1}$, we have that $\lambda^{-1} = \mu \in \text{sp}_{\mathcal{A}}(a^{-1})$. As pointed out above, this implies that $\lambda \in \text{sp}_{\mathcal{A}}(a)$. Thus, $\mu = \lambda^{-1}$ for some $\lambda \in \text{sp}_{\mathcal{A}}(a)$, which shows that the reverse inclusion also holds.

b) Let $\mathcal{B}$ be unital Banach algebra and assume that $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra-isomorphism.

Then we have $\text{sp}_{\mathcal{A}}(a) = \text{sp}_{\mathcal{B}}(\phi(a))$ for all $a \in \mathcal{A}$:

We denote both the units of $\mathcal{A}$ and $\mathcal{B}$ by e . We first note that $\phi(e) = e$. Indeed, one readily sees that $\phi(e)$ is a unit for $\phi(\mathcal{A}) = \mathcal{B}$, so $\phi(e)$ must be equal to e , by uniqueness of the unit.

Let now $a \in \mathcal{A}$. Since $\phi(e) = e$, it follows readily that if $a' \in \mathcal{A}$, then

$$a' \in \text{GL}(\mathcal{A}) \Leftrightarrow \phi(a') \in \text{GL}(\mathcal{B}).$$

So for $\lambda \in \mathbb{C}$ we get that

$$\lambda \in \text{sp}_{\mathcal{A}}(a) \Leftrightarrow \lambda e - a \notin \text{GL}(\mathcal{A}) \Leftrightarrow \phi(\lambda e - a) \notin \text{GL}(\mathcal{B}) \Leftrightarrow \lambda e - \phi(a) \notin \text{GL}(\mathcal{B}) \Leftrightarrow \lambda \in \text{sp}_{\mathcal{B}}(\phi(a)),$$

which shows the assertion.

Exercise 37 [= Exercise 4.1.3 in Pedersen's book]

Let a and b be elements in a complex, unital Banach algebra $\mathcal{A}$. Then we have that

$$\text{sp}_{\mathcal{A}}(ab) \setminus \{0\} = \text{sp}_{\mathcal{A}}(ba) \setminus \{0\}.$$

Assume $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $\lambda \notin \text{sp}_{\mathcal{A}}(ab)$. Set $c := \lambda^{-1}(e + b(\lambda e - ab)^{-1}a) \in \mathcal{A}$. Then

$$\begin{aligned} (\lambda e - ba)c &= (\lambda e - ba)\lambda^{-1}(e + b(\lambda e - ab)^{-1}a) \\ &= e + b(\lambda e - ab)^{-1}a - \lambda^{-1}ba - \lambda^{-1}bab(\lambda e - ab)^{-1}a \end{aligned}$$

Now

$$\begin{aligned} ab(\lambda e - ab)^{-1} &= (\lambda e - (\lambda e - ab))(\lambda e - ab)^{-1} \\ &= \lambda(\lambda e - ab)^{-1} - e \end{aligned}$$

Hence

$$\begin{aligned}(\lambda e - ba)c &= e + b(\lambda e - ab)^{-1}a - \lambda^{-1}ba - \lambda^{-1}b(\lambda(\lambda e - ab)^{-1} - e)a \\ &= e + b(\lambda e - ab)^{-1}a - \lambda^{-1}ba - b(\lambda e - ab)^{-1}a + \lambda^{-1}ba \\ &= e\end{aligned}$$

Similarly, we get that $c(\lambda e - ba) = e$. Thus, $\lambda e - ba \in \text{GL}(\mathcal{A})$, i.e., $\lambda \notin \text{sp}_{\mathcal{A}}(ba)$.

This shows that $\text{sp}_{\mathcal{A}}(ba) \setminus \{0\} \subseteq \text{sp}_{\mathcal{A}}(ab) \setminus \{0\}$. By symmetry, the reverse inclusion also holds, so the desired equality follows.

Exercise 38

Consider the complex Hilbert space $H = L^2([0, 1], \mathcal{B}_{[0,1]}, \mu)$, where μ denotes the Lebesgue measure on the Borel σ -algebra $\mathcal{B}_{[0,1]}$. Set $\mathcal{A} := \mathcal{B}(H)$, and let $M \in \mathcal{A}$ denote the multiplication operator given by

$$[M(g)](t) = tg(t) \quad \text{for all } g \in H \text{ and } t \in [0, 1].$$

As we have seen in a previous course, M has no eigenvalues.

We have $\text{sp}_{\mathcal{A}}(M) = [0, 1]$:

We first show that $\text{sp}_{\mathcal{A}}(M) \subseteq [0, 1]$:

Let $\lambda \in \mathbb{C} \setminus [0, 1]$. Then the function $g : [0, 1] \rightarrow \mathbb{C}$ defined by $g(t) = (\lambda - t)^{-1}$ is continuous, so the multiplication operator $G : H \rightarrow H$ associated to g is bounded. As we clearly have that $(\lambda I - M)G = G(\lambda I - M) = I_H$, we get that $\lambda \notin \text{sp}_{\mathcal{A}}(M)$. This shows that the inclusion above holds.

To show the reverse inclusion, let $\lambda \in [0, 1]$. We will show that there exists a sequence $\{\xi_n\}$ of *unit* vectors in H such that

$$\|(\lambda I_H - M)\xi_n\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This will imply that $\lambda \in \text{sp}_{\mathcal{A}}(M)$, because, otherwise, we would get that

$$1 = \|\xi_n\|_2 = \|(\lambda I_H - M)^{-1}(\lambda I_H - M)\xi_n\|_2 \leq \|(\lambda I_H - M)^{-1}\| \|(\lambda I_H - M)\xi_n\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

giving a contradiction.

Assume first that $0 \leq \lambda < 1$ and choose $N \in \mathbb{N}$ such that $[\lambda, \lambda + 1/N] \subseteq [0, 1]$. For each $n \geq N$, define $\xi_n : [0, 1] \rightarrow \mathbb{C}$ by $\xi_n = n^{1/2} \mathbf{1}_{[\lambda, \lambda + 1/n]}$. Then for each $n \geq N$ we have $\|\xi_n\|_2 = 1$ and

$$\|(\lambda I_H - M)\xi_n\|_2^2 = n \int_{\lambda}^{\lambda + 1/n} (\lambda - t)^2 dt = n \left[\frac{1}{3}(t - \lambda)^3 \right]_{t=\lambda}^{t=\lambda + 1/n} = \frac{1}{3n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We can clearly proceed similarly when $\lambda = 1$ by considering $\xi_n = n^{1/2} \mathbf{1}_{[1 - 1/n, 1]}$. Thus, in both cases, we can conclude that $\lambda \in \text{sp}_{\mathcal{A}}(M)$.

This shows that $[0, 1] \subseteq \text{sp}_{\mathcal{A}}(M)$. Altogether, the desired equality follows.

Exercise 39

Let Ω be a topological space and $\mathcal{A} = C_b(\Omega)$, i.e., $\mathcal{A}$ is the unital Banach algebra consisting of all bounded continuous complex functions on Ω , equipped with the uniform norm $\|\cdot\|_{\infty}$.

Let $f \in \mathcal{A}$. Then $\text{sp}_{\mathcal{A}}(f) = \overline{f(S)}$:

If $\lambda = f(\omega)$ for some $\omega \in \Omega$, then $(\lambda 1_{\Omega} - f)(\omega) = 0$, so $\lambda 1_{\Omega} - f \notin \text{GL}(\mathcal{A})$, i.e., $\lambda \in \text{sp}_{\mathcal{A}}(f)$.

This shows that $f(\Omega) \subseteq \text{sp}_{\mathcal{A}}(f)$. Since $\text{sp}_{\mathcal{A}}(f)$ is closed in $\mathbb{C}$, we get that $\overline{f(\Omega)} \subseteq \text{sp}_{\mathcal{A}}(f)$.

On the other hand, assume $\lambda \in \mathbb{C} \setminus \overline{f(\Omega)}$. Since $K := \overline{f(\Omega)}$ is closed, we have that $d := \inf |\lambda - z| : z \in K > 0$.

Define $g : \Omega \rightarrow \mathbb{C}$ by $g(\omega) = (\lambda - f(\omega))^{-1}$. Then

$$\|g\|_{\infty} = \sup\{|\lambda - f(\omega)|^{-1} : \omega \in \Omega\} \leq \sup\{|\lambda - z|^{-1} : z \in K\} = d^{-1} < \infty,$$

so $g \in \mathcal{A}$. Moreover, it is then clear that g is the inverse of $\lambda 1_{\Omega} - f$ in $\mathcal{A}$, i.e., $\lambda \notin \text{sp}_{\mathcal{A}}(f)$. This shows that $\text{sp}_{\mathcal{A}}(f) \subseteq \overline{f(\Omega)}$.

Altogether, we get that $\text{sp}_{\mathcal{A}}(f) = \overline{f(\Omega)}$, as desired.

Exercise 40

Let X be a complex Banach space and consider the Banach algebra $\mathcal{A} := \mathcal{B}(X)$. Let $T \in \mathcal{A}$. The adjoint operator $T^{\sharp}$ belongs then to the Banach algebra $\mathcal{B} := \mathcal{B}(X^*)$.

We have that $\text{sp}_{\mathcal{B}}(T^{\sharp}) = \text{sp}_{\mathcal{A}}(T)$:

We will first show that

$$T \in \text{GL}(\mathcal{B}(X)) \Leftrightarrow T^{\sharp} \in \text{GL}(\mathcal{B}(X^*)). \quad (1)$$

Assume $T \in \text{GL}(\mathcal{B}(X))$ and set $S = T^{-1} \in \mathcal{B}(X)$. Then we get that

$$T^{\sharp} S^{\sharp} = (ST)^{\sharp} = (I_X)^{\sharp} = I_{X^*},$$

Similarly, $S^{\sharp} T^{\sharp} = I_{X^*}$. Thus, $T^{\sharp} \in \text{GL}(\mathcal{B}(X^*))$. This shows the forward implication.

Conversely, assume $T^{\sharp} \in \text{GL}(\mathcal{B}(X^*))$. Using what we just have shown to $T^{\sharp}$, we get that $T^{\sharp\sharp} \in \text{GL}(\mathcal{B}(X^{**}))$. Let $x \rightarrow j_x$ denote the canonical isometry from X into X^{**} . Note that for $x \in X$ and $\varphi \in X^*$, we have

$$[T^{\sharp\sharp}(j_x)](\varphi) = (j_x \circ T^{\sharp})(\varphi) = j_x \circ \varphi \circ T = (\varphi \circ T)(x) = j_{T(x)}(\varphi).$$

Thus, $T^{\sharp\sharp}(j_x) = j_{T(x)}$ for every $x \in X$. This shows that if we identify X with its isometric copy inside X^{**} , then the restriction of $T^{\sharp\sharp}$ to X is equal to T . Since $T^{\sharp\sharp}$ is injective, we get that T is injective.

To show that T is surjective, we first observe that $T(X)$ is dense in X . Indeed, assume (for contradiction) that there exists some $x \in X \setminus \overline{T(X)}$. By a corollary to the second Hahn-Banach separation theorem, we can find some $\psi \in X^*$ such that $\psi(x) \neq 0$ while $\psi = 0$ on $\overline{T(X)}$. We then have $T^{\sharp}(\psi) = \psi \circ T = 0$, which implies that $\psi = 0$ since $T^{\sharp}$ is invertible in $\mathcal{B}(X^*)$, giving a contradiction.

Next, we observe that $T(X)$ is closed in X . Indeed, let $\{x_n\}$ be a sequence in X such that $T(x_n) \rightarrow y$ for some $y \in X$. For any $m, n \in \mathbb{N}$, we then have

$$\|x_n - x_m\| = \|(T^{\sharp\sharp})^{-1}(T(x_n - x_m))\| \leq \|(T^{\sharp\sharp})^{-1}\| \|T(x_n) - T(x_m)\|.$$

Since $\{T(x_n)\}$ is Cauchy, we get that $\{x_n\}$ is Cauchy. As X is complete, $x_n \rightarrow x$ for some $x \in X$, and we then get that $y = \lim_n T(x_n) = T(x)$, i.e., $y \in T(X)$, as desired.

Altogether, we get $T(X) = \overline{T(X)} = X$, i.e., T is surjective.

Thus we have shown that T is bijective. By the open mapping theorem, we get that $T \in \text{GL}(\mathcal{B}(X))$. This finishes the proof that (1) holds.

Let now $\lambda \in \mathbb{K}$. Using (1) we get

$$\begin{aligned} \lambda \in \text{sp}_{\mathcal{A}}(T) &\Leftrightarrow (\lambda I_X - T) \notin \text{GL}(\mathcal{B}(X)) \Leftrightarrow (\lambda I_X - T)^{\sharp} \notin \text{GL}(\mathcal{B}(X^*)) \\ &\Leftrightarrow (\lambda I_{X^*} - T^{\sharp}) \notin \text{GL}(\mathcal{B}(X^*)) \Leftrightarrow \lambda \in \text{sp}_{\mathcal{B}}(T^{\sharp}), \end{aligned}$$

which shows that $\text{sp}_{\mathcal{A}}(T) = \text{sp}_{\mathcal{B}}(T^{\sharp})$.