

MAT4460 H25 ORAL EXAM PART I

This is the first part of oral exam on Friday December 12, 2025. Please prepare your presentation for up to 15 minutes. You can look at your notes for this part.

Problem. Consider the discrete Heisenberg group

$$\Gamma = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\},$$

and let $\mathcal{A}$ be its group C*-algebra $C^*(\Gamma)$. We will write $g_{a,b,c}$ for the element of Γ as in the above defining formula, and set $u = g_{1,0,0}$, $v = g_{0,1,1}$, and $z = g_{0,0,1}$. (You can freely use that z generates the center of Γ .)

- (1) Write $Z \in \mathcal{A}$ for the element represented by z , and $\mathcal{I} \subset \mathcal{A}$ for the closed two-sided ideal generated by $Z - I$. Determine the quotient algebra $\mathcal{A}/\mathcal{I}$.
- (2) Let π be an irreducible representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that $\pi(Z) \neq \text{Id}_{\mathcal{H}}$. Show that the image of $\mathcal{A}$ under the homomorphism $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is generated by two unitary elements satisfying the relation of a noncommutative torus algebra. (Hint: Compute $uvu^{-1}v^{-1}$.)
- (3) What is possible as the dimension of irreducible representation for $\mathcal{A}$?
- (4) This is a soft question, but try to see if you can make sense of it: If we allow the parameters a, b, c to be real numbers, we get the usual (continuous) Heisenberg group G , and Γ sits inside G as a discrete subgroup. What kind of analogy is there between the quotient space G/Γ and the C*-algebra $\mathcal{A}$?