

MAT4460 MANDATORY ASSIGNMENT

Prepare your solutions as a PDF file and submit it via **Canvas** system. The deadline is 31 October.

Problem 1. Let $\mathcal{T} \subset \mathcal{B}(\mathcal{H}^2(\mathbb{T}))$ be the Toeplitz algebra. Show that its defining representation (the inclusion map $\mathcal{T} \rightarrow \mathcal{B}(\mathcal{H}^2(\mathbb{T}))$ regarded as a representation) is irreducible.

Hint: Use $\mathcal{K} \subset \mathcal{T}$ to estimate the commutant, and relate it to the irreducibility.

Problem 2. Let q be a real number satisfying $0 \leq q < 1$. Consider the operator $Z_q \in \mathcal{B}(\ell^2(\mathbb{N}))$ defined by

$$Z_q(e_n) = \sqrt{1 - q^{n+1}} e_{n+1} \quad (n = 0, 1, \dots).$$

- (1) Show that Z_q satisfies the relation $I - Z_q^* Z_q = q(I - Z_q Z_q^*)$.
- (2) Show that the C^* -algebra generated by Z_q is equal to $\mathcal{T}$ up to the identification $\mathcal{H}^2(\mathbb{T}) \simeq \ell^2(\mathbb{N})$ given by $z^n \leftrightarrow e_n$.

Hint: First show that $Z_q - T_z$ is compact. Then compare the ‘monomials’ T_{z^k} and Z_q^k for $k \geq 2$, or $T_{z^{-k}}$ and Z_q^{*k} for $k \geq 1$.

Problem 3. Let V be an isometry on a Hilbert space $\mathcal{H}$, and set $d = \ker V^*$. The *Wold decomposition* says that there is a Hilbert space $\mathcal{H}'$, a unitary operator U on $\mathcal{H}'$, a unitary operator $W: \mathcal{H} \rightarrow \ell^2(\mathbb{N})^{\oplus d} \oplus \mathcal{H}'$ such that

$$WVW^* = (S \oplus S \oplus \dots \oplus S) \oplus U,$$

where S is the unilateral shift operator $e_n \mapsto e_{n+1}$. Show that this implies the existence of a homomorphism of C^* -algebras $\mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ such that $T_z \mapsto V$.

Hint: Up to the identification $\mathcal{H}^2(\mathbb{T}) \simeq \ell^2(\mathbb{N})$ as above, T_z corresponds to S , which takes care of $S \oplus \dots \oplus S$. To define a homomorphism $\mathcal{T} \rightarrow \mathcal{B}(\mathcal{H}')$ mapping T_z to U , first consider the standard homomorphism $\mathcal{T} \rightarrow C(S^1)$.

Problem 4. For each $n = 1, 2, \dots$, consider the positive compact operator $P_n \in \mathcal{B}(\mathcal{H}^2(\mathbb{T}))$ defined by

$$P_n z^k = \begin{cases} z^k, & 0 \leq k \leq n, \\ (2 - \frac{k}{n}) z^k, & n < k \leq 2n, \\ 0, & k > 2n. \end{cases}$$

- (1) Compute the effect of $T_z P_n - P_n T_z$ on z^k .
- (2) Conclude that $\|T_z P_n - P_n T_z\|$ converges to 0.
- (3) Show that any $*$ -polynomial in T_z also has the same interaction with P_n .
- (4) Show that the sequence $(P_n)_{n=1}^\infty$ is a quasi-central approximate unit for the ideal $\mathcal{K} \subset \mathcal{T}$.