

Def. of C^* -alg : Davidson p.1 / Strung § 2.1

Ex. $\mathcal{H}$: (complex) Hilb. sp.

$$\leadsto \mathcal{B}(\mathcal{H}) = \{ A : \mathcal{H} \rightarrow \mathcal{H} \mid \text{bdd lin. map} \}$$

norm : op. norm $\|A\| = \sup_{x \in \mathcal{H}, \|x\| \leq 1} \|Ax\|$

invol. : A^* characterized by $(A^*x, y) = (x, Ay)$
 $x, y \in \mathcal{H}$

How to check $\|A^*A\| = \|A\|^2$

- $\|B\| = \sup_{\|x\|, \|y\| \leq 1} |(Bx, y)|$ from $\|z\| = \sup_{\|y\| \leq 1} |(z, y)|$

- $|(A^*Ax, y)|$ is big $\sim Ax$ and Ay are lin. indep.
 $= |(Ax, Ay)|$ $\|Ax\|$ big ($\sim \|Ay\|$)

Ex. X locally compact Hausdorff sp.

$\leadsto C_0(X) = \{ f : X \rightarrow \mathbb{C} \mid \text{cont.}, |f(x)| < \varepsilon \text{ outside cpt set } K_\varepsilon \subset X \}$

norm : ∞ -norm $\|f\| = \sup_{x \in X} |f(x)|$

invol : $f^*(x) = \overline{f(x)}$.

Why are C^* -algebras good?

- Flexible enough to include examples from
 - theory of differential eqs
 - groups acting on geom. objs

- Structured enough to get nice properties/invariants

→ - classification of certain C^* -algs.

- homomorphism $A \xrightarrow{f} B$ of C^* -algs

have a closed img ($\leadsto f(A)$ is a C^* -alg)

Non-examples M_n , $\|A\| = \max_{i,j} \|A_{ij}\|$, $\ell^1(G)$

How to find C^* -algs

1. as alg. of ops. on a Hilbert space

1-1. Find / make a Hilb. sp. $\mathcal{H}$

.. as L^2 -funcs $f : \int_X |f(x)|^2 d\mu(x)$ on some
measure sp. (X, μ)

.. as $\mathbb{C}^d$ f.d.

1-2. Find an interesting class of lin. transforms
on $\mathcal{H}$.

- mult. by funcs. on X

- translate points of X $\dots (\alpha \in \mathbb{C})$

1-3 Take closure for $A+B, \alpha A, AB, A^*$
and $A_n \rightarrow A_\infty$ ($\|A_n - A_\infty\| \rightarrow 0$)

2. as abstract alg. with generators, relations

2-1 Decide generators & rels.

ex. group G $g^* = g^{-1}$

2-2 Take closure for $A+B, \alpha A, AB, A^*$

2-3 Decide norm for these (s.t. $\|A^*A\| = \|A\|^2$)

2-4 Take closure for $A_n \rightarrow A_\infty$

How to talk about spectrum of elements in a
 C^* -algebra

Def. Suppose $\mathcal{A}$ is an alg (over $\mathbb{C}$) with unit

$\hookrightarrow I \cdot A = A = A \cdot I$ for some $I \in \mathcal{A}$

The spectrum of $A \in \mathcal{A}$ is

$$\sigma(A) = \{ \lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible} \}$$

Intuition $\mathcal{Q} = \mathcal{C}(X)$ for some cpt sp. X

$\leadsto \sigma(f) = \text{range of } f$

What do we do when $\mathcal{Q}$ does not have a unit?

Ex. $\mathcal{Q} = \mathcal{C}_0(X)$ for loc. cpt. but non-cpt. X .

$\mathcal{Q} = \mathcal{K}(\mathcal{H})$ compact operators

(closure of fin. rank ops.)

for ∞ -dim Hilb. sp. $\mathcal{H}$

we want $\text{ran}(f)$ or $\{\text{eigenvals of } A\}$
as spectrum of A .

$\leadsto$ Unitization (Davidson Prop I.1.3

$\tilde{\mathcal{Q}}$

Strung 2.2.2).

Idea : If $\mathcal{Q}$ does not have unit
add a new elem. I' that becomes
the unit.

$$\leadsto \tilde{\mathcal{Q}} = \{A + \lambda I' \mid A \in \mathcal{Q}, \lambda \in \mathbb{C}\}$$

use $A \lambda I' = \lambda A = \lambda I' A$ to make

sense of prod

Norm $\|A + \lambda I'\|$: norm of transf. $\mathcal{Q} + \mathcal{Q}$
 $B \mapsto (A + \lambda I')B$

(If this transf. is zero, we have $\lambda B + \lambda B = 0$
for all $B \in \mathcal{Q}$ $\leadsto -\lambda^{-1}A$ would be a unit of $\mathcal{Q}$)