

Recap C^* -algs

$$\|A^*A\| = \|A\|^2$$

- spectrum $\sigma(A)$
- unitization $\tilde{\mathcal{A}}$

commutative C^* -algebras (review) $\mathcal{A}$ comm. C^* -alg $\leadsto$ "spectrum" of $\mathcal{A}$: $\mathcal{M}_{\mathcal{A}} = \{ \varphi : \mathcal{A} \rightarrow \mathbb{C} \mid \text{nonzero hom of } C^*\text{-algs} \}$

- loc. cpt. by the weak top. for pairing with $\mathcal{A}$

 $(\varphi_i \rightarrow \varphi \text{ means } \forall A \in \mathcal{A} \quad \varphi_i(A) \rightarrow \varphi(A))$

- Gelfand transf. $\mathcal{A} \rightarrow C_0(\mathcal{M}_{\mathcal{A}})$ is an isomorphism $A \mapsto f_A$; $f_A(\varphi) = \varphi(A)$

How to get a comm C^* -alg.1. as $C_0(X)$ for some loc. cpt. sp. X 2. Start ^{unital} from noncomm. C^* -alg $\mathcal{A}$,2-1. take a normal elem $A \in \mathcal{A}$.
 $A^*A = AA^*$ 2-2. take the $*$ -polynomials of A
(polynoms. of A and A^*)2-3. take the closure for $B_n \rightarrow B_{\infty}$ (norm) $\leadsto$ commutative C^* -subalg $C^*(A) \subset \mathcal{A}$.Ex. $A \in \mathcal{A}$ normal $\leadsto \mathcal{M}_{C^*(A)}$ homeo. to $\sigma(A)$ concretely: $\mathcal{M}_{C^*(A)} \rightarrow \mathbb{C}$, $\varphi \mapsto \varphi(A)$
has img $\sigma(A)$

Davidson Cor I.3.2, Strung Th'm 2.4.1.

We can talk about $f(A) \in \mathcal{A}$ for $f \in C(\sigma(A))$
continuous functional calculus

Positive elements Davidson p.2, Strung 3.1.1

$$B = B^*, \quad \sigma(B) \subset [0, \infty).$$

Another viewpoint: $B = A^*A$ for some $A \in \mathcal{Q}$.

Why A^*A is positive (D. I.4.5, S. 3.1.10)

- $(A^*A)^* = A^*A$ from $(ST)^* = T^*S^*, S^{**} = S$

- check $\sigma(B) \subset [0, \infty)$ for $B = A^*A$

1. write $B = B_+ - B_-$, $B_{\pm} = f_{\pm}(B)$, $\frac{f_+}{0} \searrow \frac{f_-}{0}$

and set $C = \sqrt{B_-}$.

2. check $AC = 0$ (setting $T = AC$)
via both T^*T and $-T^*T$ positive

3. $B_- = 0$ from $B_+C = 0$

$$B_-^2 = CB_-C = -CA^*AC (= -T^*T)$$

Order relation among self adjoint elements

A, B self adj ($A = A^*$).

define $A \leq B$ as $B - A$ positive
($= C^*C$)

Davidson around I.4.6, Strung 3.1.11

• $A \leq B \Rightarrow X^*AX \leq X^*BX$ for any $X \in \mathcal{Q}$
use: $B - A = C^*C$

• $A \leq B$, invertible $\Rightarrow B^{-1} \leq A^{-1}$

1. $\|A^{1/2}B^{-1/2}\| \leq 1$ from $B^{-1/2}AB^{-1/2} \leq B^{-1/2}BB^{-1/2}$

$$\leadsto \|B^{-1/2}AB^{-1/2}\| \leq 1.$$

2. use $A^{1/2} B^{-1} A^{1/2} \leq I$, mult. by $A^{-1/2}$ from both sides $\Rightarrow B^{-1} \leq A^{-1}$.

What we used: positive elem C^*C satisfies $\|C^*C\| \leq 1$ iff $C^*C \leq I$.



Caution: $A \leq B$ does not imply $A^2 \leq B^2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = A + \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ in } \mathcal{M}_2$$

Next: approximate units

suppose $\mathcal{A}$ is a non-unital C^* -alg.

can we have an approx. version of unit?

1. $\mathcal{A} = K(\mathcal{H}) \hookrightarrow \mathcal{H}$ separable Hilb. sp.

1-1. choose an incr. seq. of fin. dim subsp.
 $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots \quad \bigcup_{n=1}^{\infty} \mathcal{H}_n$ dense in $\mathcal{H}$.

P_n : orth proj. to $\mathcal{H}_n$. ; P_n fin. rk.

$$\|P_n A - A\| \rightarrow 0, \quad \|A P_n - A\| \rightarrow 0.$$

1-2. index by fin. dim sp. $\mathcal{E} \subset \mathcal{H}$.

$$P_{\mathcal{E}} = \text{proj to } \mathcal{E} \Rightarrow \|P_{\mathcal{E}} A - A\| \rightarrow 0 \quad \mathcal{E} \text{ becomes big.}$$

2 $\mathcal{A} = C_0(X)$ loc. cpt. X

2-1 choose an incr. seq. of cpt subsp. $K_n \subset X$

s.t. $X = \bigcup_{n=1}^{\infty} K_n$, choose f_n s.t. $\begin{cases} 0 \leq f_n(x) \leq 1 \\ f_n(y) = 1 \quad (y \in K_n) \end{cases}$

$$\|f_n \cdot f - f\| \rightarrow 0$$

2-2 same for all cpt $K \subset X$

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