

Recap - positive elems $A = A^*$, $\sigma(A) \subset (0, \infty)$

$$\Leftrightarrow A = B^*B$$

- order rel. for self adj. $A \leq B : B - A$ pos.

Approximate identity (Davidson p.11 / Strung 3.2.1)

$(E_\lambda)_{\lambda \in \Lambda}$ for Λ : directed set

- we have partial ord. $\lambda \leq \mu$

- $\lambda, \mu \rightsquigarrow \exists \nu : \lambda, \mu \leq \nu$

$\forall A \in \mathcal{Q} : E_\lambda A \rightarrow A$ $\lambda \leq \mu \rightsquigarrow E_\lambda \leq E_\mu$

How should we choose the index set Λ ?

- if $\mathcal{Q}$ is separable we can take $\Lambda = \mathbb{N}$

• $\mathcal{K}(H)$ for sep. Hilb. sp. (countable basis)

• $C_0(X)$ for metrizable X . $\|B\| \leq 1$

- in general : $\Lambda = \{B \in \mathcal{Q} \mid \text{positive}\}$ works

(D. Thm I.4.8 / S. Thm 3.2.3) with $E_B = B$

This means : $\forall A, \varepsilon > 0 \exists B \in \mathcal{Q}, \text{pos.}$

(Key cond.) $\forall C \in \mathcal{Q}$ s.t. $B \leq C$ $\|A - CA\| \leq \varepsilon$

Part 1. $\mathcal{Q}_+^1 = \Lambda$ above is directed.

$\forall B, C \in \mathcal{Q}_+^1 \Rightarrow \exists D \in \mathcal{Q}_+^1 : B, C \leq D$

1-1 $f(t) = \frac{t}{1-t}, g(t) = \frac{t}{1+t}$ mutual inverses (for compo)

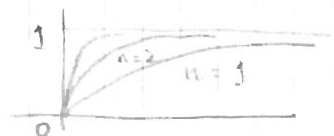
1-2 $\gamma = f(B) + f(C) \geq f(B) \geq 0$

1-3 func. calc. with $g(t)$ preserves order

$g(t) = 1 - \frac{1}{1+t}$, reduce to $A \leq B \Rightarrow B^{-1} \leq A^{-1}$

1-4 $g(\gamma) \underset{1-3}{\geq} g(f(B)) \underset{1-1}{=} B$

Part 2. finding B in the key cond.

$g(nt)$

 $\xrightarrow{\text{fixed } R} \sup_{0 \leq t \leq 1} \|g(nt)t - t\| \rightarrow 0 \quad n \rightarrow \infty$

concav. $g(nt)t - t = -\frac{t}{1+nt}$

$\leadsto$ Set $B = g(nAA^*)$ so that $\sup_{0 \leq t \leq 1} \|g(nt)t - t\| < \varepsilon^2$

Part 3 Check $C \geq B \Rightarrow \|CA - A\| \leq \|BA - A\| < \varepsilon$

3-1 $\|CA - A\|^2 \leq \|A^*(I - C)A\|$

- left hand side = $\|(CA - A)^*(CA - A)\|$
 $= \|A^*(I - C)^2 A\|$

- $(I - C)^2 \leq I - C$ from $C \geq 0, \|C\| < 1$

- $0 \leq A^*(I - C)^2 A \leq A^*(I - C)A$ from last time.

- ineq. of pos. elems $\Rightarrow$ ineq. of norms.
 (D. Lem. I.4.3, $\|x\|$ is the best const. $X \leq \|X\|I$)

3-2 $\|A^*(I - C)A\| \leq \|A^*(I - B)A\|$

use $I - C \leq I - B$

3-3 $\|A^*(I - B)A\| < \varepsilon$ from $B = g(nAA^*)$ ($n \gg 0$)

use $g(nAA^*)A = A g(nA^*A)$ $\square$

Ideals in C^* -algs $\mathcal{I} \subset \mathcal{A}$

- two-sided ideal: $A \in \mathcal{I}, B \in \mathcal{A} \Rightarrow AB, BA \in \mathcal{I}$

- norm-closed: $A_n \in \mathcal{I}, A \in \mathcal{A}, \|A_n - A\| \rightarrow 0$
 $\Rightarrow A \in \mathcal{I}$

(D. p 12, S. 1.3.1)

Where ideals appear.

- $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) = \{A: \mathcal{H} \rightarrow \mathcal{H} \mid \text{bdd lin. op.}\}$
cpt ops.
- $Y \subset X$ _{loc. cpt} closed subset $\leadsto \overset{\mathcal{CL}_Y}{\{f \in C_0(X) \mid f(y) = 0 \text{ for } y \in Y\}}$
- X mfd. $\mathcal{Q}$: order 0 pseudo-diff ops on $L^2(X)$
 $\mathcal{Q}$: pseudo-diff ops of neg. order.

$\leadsto$ leads to the quotient algebra

- $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$ Calkin algebra.
captures Fredholm operators
- $C_0(X) / \mathcal{CL}_Y \cong C_0(Y)$
- $\mathbb{F}DO_0(X) / \mathbb{F}DO_{<0}(X)$ alg of principal symbols.
(functions on the spheres of cotang. space.)



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