

Recap.

- approximate unit $\exists \lambda A \rightarrow A$ in norm
- ideal and quotient $\mathcal{I} \subset \mathcal{A}$

Ideal $\mathcal{I} \subset \mathcal{A} \leadsto$ quotient $\mathcal{A}/\mathcal{I}$ again C^* -alg.

What do we need?

- $\mathcal{A}/\mathcal{I}$ is a Banach space.
- $\mathcal{I}$ is closed under $A \mapsto A^*$
 $\leadsto$ involution on $\mathcal{A}/\mathcal{I}$ by $A_{\mathcal{I}}^* = (A^*)_{\mathcal{I}}$.
 $A_{\mathcal{I}} \in \mathcal{A}/\mathcal{I}$ image of $A \in \mathcal{A}$.

Norm on $\mathcal{A}/\mathcal{I}$.

$$\|A_{\mathcal{I}}\| = \inf_{B \in \mathcal{A}, B_{\mathcal{I}} = A_{\mathcal{I}}} \|B\| = \inf_{C \in \mathcal{I}} \|A + C\|$$

- why norm? $\|A_{\mathcal{I}}\| = 0$ means $\forall \varepsilon \exists C \in \mathcal{I}$ s.t. $\|A - C\| < \varepsilon \leadsto A \in \mathcal{I}$.
 $\mathcal{I}$ is norm closed

- why is $\mathcal{A}/\mathcal{I}$ complete?

Cauchy seq. $(A_{\mathcal{I}}^{(n)})_{n=1}^{\infty}$ in $\mathcal{A}/\mathcal{I}$ can belifted to a Cauchy seq. $(B^{(n)})_{n=1}^{\infty}$

(Ex. assum $\|A_{\mathcal{I}}^{(n)} - A_{\mathcal{I}}^{(n+1)}\| \leq \frac{1}{4^n}$)

then after knowing $B^{(n)}$ choose $B^{(n+1)}$

s.t. $(B^{(n)} - B^{(n+1)})_{\mathcal{I}} = A_{\mathcal{I}}^{(n)} - A_{\mathcal{I}}^{(n+1)}$.

$$\|B^{(n)} - B^{(n+1)}\| \leq \frac{1}{2^n}$$

Algebra structure on $\mathcal{A}/\mathcal{I}$.

$$A_{\mathcal{I}} B_{\mathcal{I}} = (AB)_{\mathcal{I}}, \quad A_{\mathcal{I}} + B_{\mathcal{I}} = (A+B)_{\mathcal{I}}$$

Invol. $A_{\mathcal{I}}^* = (A^*)_{\mathcal{I}}$.

We want: $\|A_{\mathcal{I}}^* A_{\mathcal{I}}\| = \|A_{\mathcal{I}}\|^2$

Key obs. $\mathcal{D}$ is also a C^* -alg. D. Low I.3.1, S. proof of Cor 3.2.10

hence $\exists$ approx. unit $(E_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{D}$.

Proof. We want $\mathcal{D} = \mathcal{D}^*$ ($\Leftrightarrow \mathcal{D} = \mathcal{D} \cap \mathcal{D}^*$)

Step 1 $\mathcal{D} \cap \mathcal{D}^*$ is a closed subalg of $\mathcal{D}$, inv. under involution $\leadsto C^*$ -algebra.

Step 2 take an approx. unit $(E_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{D} \cap \mathcal{D}^*$

$\leadsto$ For any $A \in \mathcal{D}$ $\|A - A E_\lambda\| \rightarrow 0$ ($\lambda \rightarrow \infty$)

$$\|A - A E_\lambda\|^2 = \underbrace{\|(A - A E_\lambda)^*\|}_{\text{in } \mathcal{D}^*} \underbrace{\|(A - A E_\lambda)\|}_{\text{in } \mathcal{D}} \rightarrow 0$$

Step 3 $A E_\lambda \in \mathcal{D} \cap \mathcal{D}^*$, hence $A \in \mathcal{D} \cap \mathcal{D}^*$ $\square$

Prop $\|A_{\mathcal{D}}\| = \lim_{\lambda \rightarrow \infty} \|A - A E_\lambda\|$ for any approx. unit $(E_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{D}$.

Proof. Enough to check " λ big $\Rightarrow \|A - A E_\lambda\| < \|A_{\mathcal{D}}\| + \varepsilon$ ".

Take $C \in \mathcal{D}$ s.t. $\|A + C\| \leq \|A_{\mathcal{D}}\| + \frac{\varepsilon}{2}$

Take λ big enough $\|C(I - E_\lambda)\| < \frac{\varepsilon}{2}$

$$A - A E_\lambda = \underbrace{(A + C)(I - E_\lambda)}_{\text{pos. norm} \leq 1} - \underbrace{C(I - E_\lambda)}_{\text{norm} < \frac{\varepsilon}{2}}$$

take norm to bound by $\|A + C\| \cdot 1 + \frac{\varepsilon}{2}$ $\square$

Thm (D. I.3.4 / Strung Exercise 3.4.11)

The norm on $\mathcal{Q}/\mathcal{D}$ satisfies C^* -condition

$$\begin{aligned} \text{proof } \|A_{\mathcal{D}}^* A_{\mathcal{D}}\| &= \lim_{\lambda} \|A^* A (I - E_\lambda)\| \\ &\geq \lim_{\lambda} \underbrace{\|(I - E_\lambda) A^* A (I - E_\lambda)\|}_{\text{pos. norm} \leq 1} = \lim_{\lambda} \|A(I - E_\lambda)\|^2 \\ &= \|A_{\mathcal{D}}\|^2. \end{aligned}$$

Converse ineq. $A_{\mathcal{D}} = B_{\mathcal{D}} \Rightarrow A_{\mathcal{D}}^* A_{\mathcal{D}} = (B^* B)_{\mathcal{D}}$
 $\Rightarrow \|A_{\mathcal{D}}^* A_{\mathcal{D}}\| \leq \|B^* B\| = \|B\|^2 \sim \|A_{\mathcal{D}}\|^2$

Thm (D. I. 5.5, S. Ex. 3.4.11)

($\pi : \mathcal{A} \rightarrow \mathcal{B}$ nonzero hom. of C^* -algebra

1. Then the image of π is norm-closed

2. If π is injective, it is isometric. $\|A\| = \|\pi(A)\|$

Proof. 2-1. For positive A , $\|A\|$ is the best const. s.t. $A \leq \|A\|I$. (in the unitization)

π extends to unitization by $\pi(I) = I$.

$\leadsto$ we have the same characterization for

$\pi(A)$ ($\pi(A) \leq \|A\| \cdot I$, best estim.)

(2-2 For general A , consider A^*A .

1. $\mathcal{I} = \ker \pi$ is an ideal of $\mathcal{A}$.

$\leadsto \mathcal{A}/\mathcal{I}$ is a C^* -alg,

injective hom $\pi_{\mathcal{I}} : \mathcal{A}/\mathcal{I} \rightarrow \mathcal{B}$,
 $A + \mathcal{I} \mapsto \pi(A)$ $\square$

Hereditary subalgebra (D. p. 13 / S. p. 33)

C^* -subalg $\mathcal{B} \subset \mathcal{A}$ is hereditary if

$B \in \mathcal{B}$ is pos., $A \in \mathcal{A}$ is pos. and $A \leq B \Rightarrow A \in \mathcal{B}$.

Ex. If $P \in \mathcal{A}$ is a projection. $P = P^* = P^2$

then $P\mathcal{A}P$ is hereditary.

$$(I-P)B(I-P) = 0$$

$$0 \leq (I-P)A(I-P) \leq (I-P)B(I-P) \leq 0$$

Ex. ideal $\mathcal{I} \subset \mathcal{A}$ is hereditary

