

Operator topologies on $\mathcal{B}(\mathcal{H})$

(Davidson I.6 / Strung Def 5.1.1)

Topology on $\mathcal{H} \rightsquigarrow$ topology on $\mathcal{B}(\mathcal{H})$ by"if $U \subset \mathcal{H}$ is open, and $x \in \mathcal{H}$, then $V_{U,x} = \{T \in \mathcal{B}(\mathcal{H}) \mid Tx \in U\}$ should be open"Topologies on $\mathcal{H}$

- norm topology $\rightsquigarrow$ we consider $U_{y,\varepsilon} = \{z \in \mathcal{H} \mid \|y-z\| < \varepsilon\}$
for $y \in \mathcal{H}$, $\varepsilon > 0$

- weak topology $\rightsquigarrow$ we should consider

$$U_{y,\alpha,\varepsilon} = \{z \in \mathcal{H} \mid |(z,y) - \alpha| < \varepsilon\}$$

for $y \in \mathcal{H}$, $\alpha \in \mathbb{C}$, $\varepsilon > 0$ $\rightsquigarrow$ strong / weak operator topology on $\mathcal{B}(\mathcal{H})$ we write $T_\alpha \xrightarrow{\text{SOT}} T$, $T = \text{SOT-}\lim_{\alpha \in \Lambda} T_\alpha$ etc

so $\begin{cases} T_\alpha \xrightarrow{\text{SOT}} T \text{ means } \forall x \quad T_\alpha x \rightarrow Tx \text{ in norm} \\ T_\alpha \xrightarrow{\text{WOT}} T \text{ means } \forall y, x \quad (T_\alpha x, y) \rightarrow (Tx, y) \\ (\forall x \quad T_\alpha x \rightarrow Tx \text{ in wk. top.}) \end{cases}$

Why do we consider these topologies?

they give analogue of convergence in measure

 $\rightsquigarrow$ give analogues of L^∞ -functions when we

start from "polynomials" / "continuous functions"

$\rightsquigarrow$ get many orthogonal projections ($\leftrightarrow$ characteristic functions of measurable sets) as limits

Example. $C_0(\mathbb{R}) \subset B(L^2(\mathbb{R}))$ realized as pointwise multiplication $f \in C_0(\mathbb{R}) \mapsto T_f \in B(L^2(\mathbb{R}))$ defined by $(T_f g)(t) = f(t)g(t)$ $g \in L^2(\mathbb{R})$

$\rightarrow$ wot/sot closure gives $L^\infty(\mathbb{R})$

Example $\mathcal{K}(\mathcal{H}) \subset B(\mathcal{H})$ is dense in sot/wot

Price to pay

- more nets converge to 0 than the norm top.

$f_n(t) = e^{2\pi i n t} \in L^\infty(\mathbb{R}) \mapsto T_n = T_{f_n}$ conv. to 0 in the wot $(f_n g, h) \rightarrow 0$.

- composition of operators is not continuous

$f_n \xrightarrow{\text{wot}} 0$, $\bar{f}_n \xrightarrow{\text{wot}} 0$ but $f_n \bar{f}_n = 1$ ($\forall n$)

Other sensible topologies: 1. Strong* - topology

$T_\alpha \xrightarrow{s^*} T$ means $T_\alpha \xrightarrow{\text{sot}} T$ and $T_\alpha^* \xrightarrow{\text{sot}} T^*$

2. σ -weak topology: weak op. top. for the inclusion $B(\mathcal{H}) \subset B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$

$T \mapsto T \otimes \text{id}_{\ell^2(\mathbb{N})}$ trace pairing $\downarrow$

functionals of the form $T \mapsto \sum_{n=1}^{\infty} (Tx_n, y_n)$

for $x_n, y_n \in \mathcal{H}$ s.t. $\sum \|x_n\|^2 < \infty$, $\sum \|y_n\|^2 < \infty$

become continuous for the σ -wk top

Rem. WOT is defined as the weak top. for the collection of functionals $B(\mathcal{H}) \rightarrow \mathbb{C}$, $T \mapsto (Tx, y)$

Any WOT-contin. functional is a linear comb. of these

Prop (D. I. 6.1 / S. 5.2.8)

If $f: \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ is cont. for the SOT,

it is of the form $f(T) = \sum_{i=1}^n (Tx_i, y_i)$

for some $x_i, y_i \in \mathcal{H}$ (hence WOT-cont.)

Proof. Step 1 (general topology) $V \subset \mathcal{B}(\mathcal{H})$ neighborhood of 0 for the SOT $\Rightarrow$ finitely many basic

open sets $V_{x_i, \varepsilon_i} = \{T \mid \|Tx_i\| < \varepsilon_i\} \subset \mathcal{B}(\mathcal{H})$

s.t. $\bigcap_{i=1}^n V_{x_i, \varepsilon_i} \subset V$

Step 2 $\exists x_1, \dots, x_n \forall T: |f(T)| \leq \sqrt{\sum_{i=1}^n \|Tx_i\|^2}$

$V = \{T \mid |f(T)| < 1\}$ is a SOT-neighborhood of 0

$\Rightarrow$ we set $x_1, \dots, x_n$ s.t. $V_{x_i, 1} \subset V$

We have $W = \{T \mid \sum_{i=1}^n \|Tx_i\|^2 < 1\} \subset V_{x_i, 1}$ for all i

$\Rightarrow W \subset V$ this means $\sqrt{\sum_i \|Tx_i\|^2} < 1 \Rightarrow |f(T)| < 1$

Get the claim by scaling

Step 3 Let $\mathcal{H}' = \{(Tx_1, \dots, Tx_n) \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^n$

Then $F: \mathcal{H}' \rightarrow \mathbb{C}, (Tx_i)_{i=1}^n \mapsto f(T)$ is bounded, norm at most 1

Step 4 Extend F to $\mathcal{H}^n$ as a bdd func. $\tilde{F}: \mathcal{H}^n \rightarrow \mathbb{C}$

by the Riesz rep. thm. $\exists (y_i)_{i=1}^n \tilde{F}((z_i)_{i=1}^n) = \sum_{i=1}^n (z_i, y_i)$

$\Rightarrow$ When $z_i = Tx_i$ we have $f(T) = \sum_{i=1}^n (Tx_i, y_i)$ $\square$

