

Recap - strong and weak operator topologies on $\mathcal{B}(\mathcal{H})$ (pointwise convergence)

Commutant of $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ (Davidson I.7, Strung 5.1.5)

Ex. Γ (discrete) group, $\mathcal{H} = \ell^2(\Gamma)$

$\mathcal{S} = \{p_g \mid g \in \Gamma\}$ with right translation ops

$p_g e_h = e_{hg^{-1}}$ for the ONB $(e_h)_{h \in \Gamma}$

$\leadsto \mathcal{S}'$ contains left transl. ops, $\lambda_g : e_h \mapsto e_{gh}$

Prop. $\mathcal{S}'$ is closed for the WOT (hence SOT, norm closed)

Proof. $T \notin \mathcal{S}' \leadsto \exists$ WOT-neighborhood of T disjoint

with $\mathcal{S}'$.
(We want)

Take $S \in \mathcal{S}$ s.t. $TS \neq ST$, and choose

$x \in \mathcal{H}$ s.t. $TSx \neq STx$

then choose y s.t. $(TSx, y) \neq (STx, y)$

Set $W_\varepsilon = \{A \in \mathcal{B}(\mathcal{H}) \mid (Ax, y) \approx_\varepsilon (TSx, y),$
WOT-open $(Ax, S^*y) \approx_\varepsilon (Tx, S^*y)\}$

ε small to forbid $(ASx, y) = (Ax, S^*y)$

Then $A \in W_\varepsilon \Rightarrow AS \neq SA$ $\square$

Ex. Projection to invariant subspace

$\mathcal{H}_1 \subset \mathcal{H}$ closed subsp, $x \in \mathcal{H}_1, A \in \mathcal{S} \Rightarrow Ax, A^*x \in \mathcal{H}_1$

$\leadsto$ orthog. proj. $P: \mathcal{H} \rightarrow \mathcal{H}_1$ belongs to $\mathcal{S}'$

Rem. $\mathcal{S}'$ is closed under compo. $T_1 T_2 S = T_1 S T_2 = S T_1 T_2$

if $T_i \in \mathcal{S}', S \in \mathcal{S}$
 if $\mathcal{S}$ is closed under invol, same for $\mathcal{S}'$

Suppose $\mathcal{Q} \subset \mathcal{B}(\mathcal{H})$ is a subalg. closed under invol.

The double commutant $\mathcal{Q}'' = (\mathcal{Q}')'$ is (

- subalg. of $\mathcal{B}(\mathcal{H})$, closed under invol
- contains $\mathcal{Q}$ ($A \in \mathcal{Q}$ and $B \in \mathcal{Q}'$ commute)
- closed for WOT

Thm (D.I.7.1 / S.3.2.7)

Suppose $\mathcal{Q} \subset \mathcal{B}(\mathcal{H})$ is as above, and

$\mathcal{Q}\mathcal{H}$ = linear span of Ax for $A \in \mathcal{Q}, x \in \mathcal{H}$

is dense in $\mathcal{H}$ ($\mathcal{Q}$ acts nondegenerately on $\mathcal{H}$) (

Then $\mathcal{Q}''$ = closure of $\mathcal{Q}$ for SOT = same for WOT

Proof main idea: $T \in \mathcal{Q}'', x \in \mathcal{H}$

$$\Rightarrow \exists A \in \mathcal{Q} \text{ s.t. } Ax \sim Tx$$

Step 1. Set $\mathcal{H}_1$ = closure of $\mathcal{Q}x = \{Ax \mid A \in \mathcal{Q}\}$

$\leadsto \mathcal{H}_1$ is $\mathcal{Q}$ -inv., hence orth. proj. P to $\mathcal{H}_1$ belongs to $\mathcal{Q}'$

Step 2. $PTx \in \mathcal{H}_1$, hence approximated by some Ax . $PTx = T Px = Tx$.

Why is this enough? In general, we want

$$T \in \mathcal{Q}'', x_1, \dots, x_n \in \mathcal{H} \Rightarrow \exists A \in \mathcal{Q} : Tx_i \sim Ax_i$$

Consider $\mathcal{Q} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}^n)$ $\pi(A)(y_i)_{i=1}^n = (Ay_i)_{i=1}^n$

Step 4. $\pi(\mathcal{Q})' = M_n(\mathcal{Q}')$ $B = (B_{ij})_{i,j=1}^n$

acts on $\mathcal{H}^n$ by $B \cdot (y_i)_i = (\sum_j B_{ij} y_j)_{i=1}^n$

Step 5. $\pi(\mathcal{Q})'' = \pi(\mathcal{Q}'')$ so we get (

$\pi(T)(x_i)_i \sim \pi(A)(x_i)_i$ for some $A \in \mathcal{Q}$.

We say $\mathcal{Q} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra

on $\mathcal{H}$ if $\mathcal{Q} = \mathcal{Q}''$ (hence $= \overline{\mathcal{Q}}^{\text{SOT}} = \overline{\mathcal{Q}}^{\text{WOT}}$, nondeg.)

How to get a von Neumann alg.

take $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ closed under invol

$\leadsto \mathcal{Q} = \mathcal{S}'$ is a vN alg

Rem. If $\mathcal{Q} \subset \mathcal{B}(\mathcal{H})$ is a vN alg,

set $\mathcal{Q}_* = \{ \varphi: \mathcal{Q} \rightarrow \mathbb{C} \mid \varphi \text{ lin, restr. of a G-wt cont. func on } \mathcal{B}(\mathcal{H}) \}$

Then $\mathcal{Q} \simeq (\mathcal{Q}_*)^*$ as a Banach space.

Fact: Kaplansky's density thm. (D. I. 7.3)

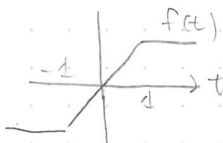
$\mathcal{Q} \subset \mathcal{B}(\mathcal{H})$ C^* -subalg, nondeg.

$\leadsto \mathcal{Q}_{\text{sa},1} = \{ A \in \mathcal{Q} \mid A = A^*, \|A\| \leq 1 \}$

is SOT-dense in $\mathcal{Q}_{\text{sa},1}''$

Idea: First find unbounded net $A_\alpha \rightarrow T \in \mathcal{Q}''$

- take $B_\alpha = \frac{1}{2}(A_\alpha + A_\alpha^*)$ to make it self adj.

- func. calc. by 

$\|f(A_\alpha)\| \leq 1$, $f(T) = T$ from $\|T\| \leq 1$

- check $A \mapsto f(A)$ is SOT-continuous

