

Recap. - commutant $\mathcal{B}'$

- double commutant theorem $\mathcal{Q}'' = \overline{\mathcal{Q}}^{wot} = \overline{\mathcal{Q}}^{sot}$

Representation of C^* -algebra

(Davidson I.9, Strung 4.2)

homomorphism $\mathcal{Q} \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$

$\hookrightarrow$ represent each $A \in \mathcal{Q}$ as an operator $\pi(A)$ on $\mathcal{H}$
compatible with prod., lin. comb., adjoint

Example. $\mathcal{T}$: Toeplitz algebra $\subset \mathcal{B}(H^2(\mathbb{T}))$

$H^2(\mathbb{T})$ L^2 -Hardy space $= \{ \sum_{n=0}^{\infty} a_n z^n \mid \sum |a_n|^2 < \infty \}$
 $\subset L^2(\mathbb{T})$

Given $f \in C(\mathbb{T})$, def. $T_f \in \mathcal{B}(H^2(\mathbb{T}))$ by

$$T_f g = P(fg), \quad P: L^2(\mathbb{T}) \rightarrow H^2(\mathbb{T})$$

orth. proj.

$T_{f_1} T_{f_2} - T_{f_1 f_2}$ is compact.

$\leadsto \mathcal{T} = \{ T_f + K \mid f \in C(\mathbb{T}), K \in \mathcal{K}(H^2(\mathbb{T})) \}$ is a
(norm-closed) subalg of $\mathcal{B}(H^2(\mathbb{T}))$

rep. $\pi: \mathcal{T} \rightarrow \mathcal{B}(L^2(\mathbb{T})), \pi(T_f + K) = \text{mult. by } f$
"take out the Toeplitz symbol"

Positive functionals. (D. p. 29, S. 4.1)

(bounded) linear map $\mathcal{Q} \xrightarrow{\phi} \mathbb{C}$ s.t. $\phi(A) \geq 0$
for positive $A \in \mathcal{Q}$

State $\equiv$ positive func. ϕ s.t. $\|\phi\| = 1$.

How do we get a pos. func. / state?

rep. $\pi: \mathcal{Q} \rightarrow \mathcal{B}(\mathcal{H})$, vector $x \in \mathcal{H}$

$\leadsto$ functional $f(A) = (\pi(A)x, x)$

- positivity $f(B^*B) \geq 0$ from $f(B^*B) = (\pi(B^*B)x, x)$
 $= (\pi(B)x, \pi(B)x)$

- x unit vector $\leadsto |f(A)| \leq \|Ax\| \cdot \|x\| \leq \|Ax\|$

Prop (D. Lem 1.9.3, S. Prop. 4.1.7)

$(E_\lambda)_{\lambda \in \Lambda}$ approx. unit of $\mathcal{A}$, ϕ pos. func.

Then $\|\phi\| = \lim \phi(E_\lambda)$

Key ingredient: Gelfand-Naimark-Segal inner product

$$\langle A, B \rangle = \phi(B^* A) \quad (\text{Hermitean inn. prod.})$$

- $\langle A, A \rangle \geq 0$ from positivity

- $\langle B, A \rangle = \overline{\langle A, B \rangle}$ from $\phi(X) \in \mathbb{R}$ for $X = X^*$
 $\hookrightarrow \phi(X_+) = \phi(X_-)$

$\leadsto$ Cauchy-Schwarz inequality

$$|\langle A, B \rangle|^2 \leq \langle A, A \rangle \cdot \langle B, B \rangle \quad \leftarrow |f(B^* A)| \leq f(A^* A) f(B^* B)$$

(follows from $\langle zA + tB, zA + tB \rangle \geq 0$ for $t \in \mathbb{R}, z \in \mathbb{C}$)

Proof of Prop.

Step 1. $\lim_{\lambda \rightarrow \infty} \phi(E_\lambda)$ exists, $\leq \|\phi\|$

$\phi(E_\lambda)$ is an incr. net of numbers in $[0, \|\phi\|]$

• $\lambda \leq \mu \Rightarrow E_\lambda \leq E_\mu$ i.e. $E_\lambda - E_\mu$ pos.

$\Rightarrow \phi(E_\lambda - E_\mu) \geq 0$ i.e. $\phi(E_\mu) \leq \phi(E_\lambda)$

• $0 \leq \phi(E_\lambda) \leq \|\phi\|$ from $\|E_\lambda\| \leq 1$.

Step 2. $\|\phi\| \leq \lim \phi(E_\lambda)$

Enough to show $|\phi(A)|^2 \leq \lim \phi(E_\lambda) \cdot \|\phi\|$ $\leftarrow$ for $\|A\| \leq 1$

$E_\lambda A \rightarrow A \leadsto \phi(E_\lambda A) \rightarrow \phi(A)$

$\leadsto$ Enough to get $|\phi(E_\lambda A)|^2 \leq \phi(E_\lambda) \cdot \|\phi\|$

Step 3. $|\phi(E_\lambda A)|^2 \leq \phi(E_\lambda) \cdot \|\phi\|$

CS ineq. $\leadsto |\phi(E_\lambda A)|^2 \leq \phi(A^* A) \cdot \phi(E_\lambda^2)$

$\|A^* A\| = \|A\|^2 \leq 1 \leadsto \phi(A^* A) \leq \|\phi\|$

cont. $\phi(E_\lambda^2) \leq \phi(E_\lambda)$ from $0 \leq E_\lambda \leq I$ $\square$

GNS representation (D. Thm I.9.6 / S. 4.2)

Pos. func. $\phi : \mathcal{A} \rightarrow \mathbb{C}$

$\leadsto$ Hilbert space $\mathcal{H}_\phi$, vector $x_\phi \in \mathcal{H}_\phi$

representation $\pi_\phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\phi)$

such that $\phi(A) = (\pi_\phi(A)x_\phi, x_\phi)$, $\|x_\phi\|^2 = \|\phi\|$

Const. of $\mathcal{H}_\phi$: take $\mathcal{A}$ with GNS inn. prod.

$\leadsto$ take quot. $\mathcal{A} / \mathcal{N}_\phi$, $\mathcal{N}_\phi = \{A \mid \phi(A^*A) = 0\}$

($\langle A, B \rangle$ defs a genuine inn. prod.)

$\leadsto$ complete as a Hilbert space

Const. of π_ϕ : write $\Lambda_\phi(A)$ for the vec of $\mathcal{H}_\phi$ represented by A .

$$\pi_\phi(A) \Lambda_\phi(B) = \Lambda_\phi(AB)$$

$$\text{boundedness } \|\Lambda_\phi(AB)\| \leq \|A\| \cdot \|\Lambda_\phi(B)\|$$

$$\text{from } B^* A^* A B \leq \|A\|^2 B^* B$$

(use $A^*A \leq \|A\|^2 \cdot I$ in unitization)

Const. of x_ϕ : take $\lim \Lambda(E_\lambda)$ in $\mathcal{H}_\phi$

Convergence of $\Lambda_\phi(E_\lambda)$:

$$\|\Lambda_\phi(E_\lambda) - \Lambda_\phi(E_\mu)\|^2 = \phi((E_\lambda - E_\mu)^2)$$

$$= \underbrace{\phi(E_\lambda^2)}_{\sim \|\phi\|} + \underbrace{\phi(E_\mu^2)}_{\sim \|\phi\|} - \underbrace{(\phi(E_\lambda E_\mu) + \phi(E_\mu E_\lambda))}_{\sim 2\phi(E_\lambda) \sim 2\|\phi\|}$$

$$\phi(A) = (\pi_\phi(A)x_\phi, x_\phi) \text{ from}$$

$$\text{right hand side} = \lim \langle \pi_\phi(A) \Lambda_\phi(E_\lambda), \Lambda_\phi(E_\lambda) \rangle$$

$$= \lim \langle \Lambda_\phi(A E_\lambda), \Lambda_\phi(E_\lambda) \rangle = \lim \underbrace{\phi(E_\lambda A E_\lambda)}_{\sim A} \quad \square$$

