

Recap Gelfand-Naimark-Segal construction

- $\langle A, B \rangle = \phi(B^*A)$ from pos. func. $\phi: \mathcal{Q} \rightarrow \mathbb{C}$
- $\pi_\phi(A)$: "left mult. by A "
- $x_\phi = \lim_{\lambda} \wedge_{\phi}(E_{\lambda})$ for approx id. $(E_{\lambda} |_{\mathcal{Q}})$ s.t. $(\pi_\phi(A) x_\phi, x_\phi) = \phi(A)$

Existence of enough states / representations

Prop. (Davidson Lem I.9.10 / Strung Prop. 4.1.10)

Given $A \in \mathcal{Q}$ positive (or selfadj.)

there is a state ϕ on $\mathcal{Q}$ s.t. $|\phi(A)| = \|A\|$.

Proof. Step 1. Subalg $\mathcal{B} \subset \tilde{\mathcal{Q}}$ (unitization)

$\mathcal{B} = C^*$ -alg generated by A, I .

unital comm. C^* -alg. $\mathcal{B} \simeq C(\sigma(A))$

Step 2 Find a state on $\mathcal{B}$

$\|A\| = \max_{\phi: \mathcal{B} \rightarrow \mathbb{C} \text{ hom}} |\phi(A)|$ from $\mathcal{B} \simeq C(\sigma(A))$

ϕ would be a state: $\phi(B^*B) = \phi(B^*)\phi(B) \geq 0$
 $\phi(I) = 1$

Step 3 Extend ϕ to $\tilde{\mathcal{Q}}$ as a state.

Any ext. $\tilde{\phi}$ as a func. of norm 1 will do.

We want positivity for $\tilde{\phi}$: $\tilde{\phi}(B^*B) \geq 0$ $B \in \mathcal{Q}$.

Set $\mathcal{C} = C^*$ -alg generated by B^*B, I .

so $\mathcal{C} \simeq C(\sigma(B^*B))$; $\tilde{\phi}$ restricts to functional of norm 1, $\tilde{\phi}(I) = 1$.

Riesz-Markov thm gives a measure on $\sigma(B^*B) \subseteq \mathbb{R}$

Con. Any C^* -alg $\mathcal{A}$ has a rep. $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$

(G-N Thm) s.t. $\ker \pi = 0$ (hence isometric, $\mathcal{A}$ can be identified with $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$).

Sketch Take $\bigoplus_{\phi} \pi_{\phi}$ for all states $\phi: \mathcal{A} \rightarrow \mathbb{C}$

Irreducible representations and pure states

Irred. rep. (D. I. 9 / S. 3.3.5) $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$

st. if $\mathcal{H}_1 \subset \mathcal{H}$ is a closed subsp. st.

$$x \in \mathcal{H}_1, A \in \mathcal{A} \Rightarrow \pi(A) \cdot x \in \mathcal{H}_1 \quad (\pi(\mathcal{A})\mathcal{H}_1 \subset \mathcal{H}_1)$$

then $\mathcal{H}_1 = \{0\}$ or $\mathcal{H}_1 = \mathcal{H}$.

Prop. π is irred. $\Leftrightarrow \pi(\mathcal{A})' = \mathbb{C}I$ in $\mathcal{B}(\mathcal{H})$.

Idea. $\pi(\mathcal{A})\mathcal{H}_1 \subset \mathcal{H}_1 \Leftrightarrow$ orth. proj. $P: \mathcal{H} \rightarrow \mathcal{H}_1$ is in $\pi(\mathcal{A})'$

- $\pi(\mathcal{A})'$ has enough projections.

Pure state (D. p. 30 / S. 3.3.1-3): extreme points

of $\{\text{states on } \mathcal{A}\} \subset \mathcal{A}^*$
convex, w^* -closed if $\mathcal{A}$ is unital

Ex. $\left\{ \begin{array}{l} \text{Lem (S. 3.3.3)} \end{array} \right.$ ϕ pure state on $\mathcal{A} \Leftrightarrow$

if ψ is a pos. func. st. $\psi(B^*B) \leq \phi(B^*B)$

then $\exists 0 \leq t \leq 1$ st. $\psi = t\phi$.

Proof Set $t = \|\psi\|$ (≤ 1 by $\|\psi\| = \lim \psi(E_{\lambda})$)

$\psi' = t^{-1}\psi$ state. (assume $t \neq 0$)

Step 1. $\phi = t\psi' + (1-t)(\phi - \psi)/\|\phi - \psi\|$

$\phi - \psi$ is pos., $1-t = \lim (1 - \psi(E_{\lambda})) = \|\phi - \psi\|$
 $\uparrow$
 $1 - \|\psi\| = \lim \phi(E_{\lambda})$

Step 2. ϕ extreme state $\leadsto \phi = \psi'$ by $t \neq 0$ &

Thm. (D. I. 9.8 / S. 5.3.8) ϕ state on $\mathcal{A}$

ϕ is pure $\Leftrightarrow$ the GNS rep $\pi_{\phi}: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_{\phi})$ is irreducible

Pure $\Rightarrow$ irred. : check $\pi_\phi(\mathcal{Q})' = \mathbb{C}I$.

Take $T \in \pi_\phi(\mathcal{Q})'$ positive, $0 \leq T \leq I$.

(this can be proj $P: \mathcal{H} \rightarrow \mathcal{H}_1$ to inv. subsp.)

Set $\psi(A) = (\pi_\phi(A) T x_\phi, x_\phi)$ for $A \in \mathcal{Q}$.

Step 1. $0 \leq \psi(B^*B) \leq \phi(B^*B)$ for $B \in \mathcal{Q}$.

$$\psi(B^*B) = (\pi_\phi(B^*) T \pi_\phi(B) x_\phi, x_\phi)$$

$$0 \leq T \leq I \leadsto 0 \leq \pi_\phi(B^*) T \pi_\phi(B) \leq \pi_\phi(B^*) \pi_\phi(B)$$

Step 2. $T = tI$ for some $0 \leq t \leq 1$.

Use Lem to get t s.t. $\psi(A) = t \phi(A)$

$$\leadsto \psi(C^*B) = t \phi(C^*B) \text{ i.e.}$$

$$(T \pi_\phi(B) x_\phi, \pi_\phi(C) x_\phi) = t (\pi_\phi(B) x_\phi, \pi_\phi(C) x_\phi)$$

$\nwarrow$ span dense subsp. of $\mathcal{H}_\phi$ $\nwarrow$ $\Lambda_\phi(B)$

$$\leadsto T = tI. \quad \square$$

Irred $\Rightarrow$ pure : suppose $\phi = t\psi + (1-t)\psi'$
 $0 < t < 1$, ψ, ψ' st

Step 1. $t\psi(A) = (\pi_\phi(A) T x_\phi, x_\phi)$ for
 some $T \in \pi_\phi(\mathcal{Q})'$ pos. $\|T\| \leq 1$.

Define T by $(T \Lambda_\phi(A), \Lambda_\phi(C)) = t\psi(C^*B)$

use $0 \leq t\psi \leq \phi$ to get $0 \leq T \leq I$

Step 2. $\pi_\phi(\mathcal{Q})' = \mathbb{C}I \Rightarrow \psi = \phi \quad \square$

Examples 1. $\mathcal{Q} = C_0(X)$ loc. cpt. X

$\phi(f) = f(x_0)$ for some fixed $x_0 \in X$

2. $\mathcal{Q} = \mathcal{B}(\mathcal{H})$, $\phi(A) = (Ax, x)$ for

some fixed unit vec. $x \in \mathcal{H}$

Riesz - Markov theorem for compact spaces

if $\phi : C(X) \rightarrow \mathbb{C}$ is a bounded functional, then $\exists$ complex measure μ

$$\text{s.t. } \phi(f) = \int_X f(x) d\mu(x)$$

$$\|\phi\| = |\mu|(X) \quad (\text{total variation.})$$

$$= \sup \sum_i |\mu(Y_i)| \quad X = \bigsqcup_i Y_i$$

Given $\|\phi\| = \phi(1_X)$ i.e. $|\mu|(X) = \mu(X)$

any measurable subset $Y \subset X$ would have $\mu(Y) \geq 0$

i.e. μ is a usual measure.

So $\phi(f) \geq 0$ for $f \geq 0$