

Quasicontral approximate units for ideals

Davidson p.35 / Strung 3.2.4

 $\mathcal{Q} \subset \mathcal{Q}$ ideal in a C^* -alg. $(E_\lambda)_\lambda \subset \mathcal{Q}$

- approx. unit of $\mathcal{Q}$. $E_\lambda A \rightarrow A$ ($A \in \mathcal{Q}$)
- approx. central in $\mathcal{Q}$.

$$\|E_\lambda A - A E_\lambda\| \rightarrow 0 \quad \text{for } A \in \mathcal{Q}$$

What can we get out of such $(E_\lambda)_\lambda$?Weyl-von Neumann-Berg theorem (D. Cor II.4.2)
separable $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ selfadj, $A_i A_j = A_j A_i$ $\Rightarrow \exists$ orthonormal basis $(e_n)_{n=0}^\infty \subset \mathcal{H}$ s.t.

$$\forall \varepsilon > 0 \quad A_i = D_i + K_i$$

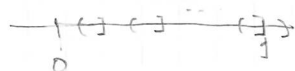
- D_i is diagonal with resp. to $(e_n)_n$.

(i.e. $D_i e_n = \lambda_{i,n} e_n$ for $\lambda_{i,n} \in \mathbb{C}$)

- K_i is a compact op., $\|K_i\| < \varepsilon$.

How to cast this into C^* -alg. problems. $\langle \text{assume } 0 \leq A_i \leq I \rangle$ $\mathcal{Q} = C^*$ -alg. generated by $A_1, \dots, A_n, I$. $\simeq C(X)$ $X \subset [0,1]^n$ "joint spectrum" of $(A_i)_i$ the i -th coord. func. $x_i \in C(X)$ corresponds to $A_i \in \mathcal{Q}$

$$X_{i,k} = \left\{ x \in X \mid \frac{2^{j-1}}{2^k} < x_i \leq \frac{2^j}{2^k} \right\}$$

for some $j = 1, \dots, 2^{k-1}$ 

$$\Rightarrow x_i = \sum_{k=1}^{\infty} 2^{-k} 1_{X_{i,k}}$$

 $\mathcal{E} = C^*$ -alg generated by projections

$$P_{i,k} = 1_{X_{i,k}}(A_1, \dots, A_n) \in \mathcal{B}(\mathcal{H})$$

so $\mathcal{Q} \subset \mathcal{E}$.

$\Sigma + \mathcal{K}(\mathcal{H})$ is a C^* -alg, $\mathcal{K}(\mathcal{H})$ ideal.

$\leadsto \exists$ quasicentral unit for this ideal

We refine this to get control on $\|k\|$.

Choose N s.t. $2^{-N} < \varepsilon$.

increasing seq. of fin. dim. subsp.

$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots \quad \bigcup_{l=1}^{\infty} \mathcal{H}_l$ dense in $\mathcal{H}$.

make $\mathcal{H}_l$ invariant under $P_{i,k}$ $k \leq l$

$\mathcal{H}'_l = \text{span of } P_{i,k}x \text{ for } x \in \mathcal{H}_l, k \leq l$
 $i=1, \dots, n$
($\supset \mathcal{H}_l$)

Q_l : orthog. proj to $\mathcal{H}'_l$. (so $Q_l \in \mathcal{K}(\mathcal{H})$)

$P_{i,k} Q_l = Q_l P_{i,k}$ for $k \leq l$. from $P_{i,k} \mathcal{H}_l \subset \mathcal{H}_l$.

Set $D_{i,k} = \begin{cases} P_{i,k} & \text{for } k \leq N \\ P_{i,k} (I - Q_k) & \text{for } k > N \end{cases}$

we have $D_{i,k} Q_l = Q_l D_{i,k}$ for $k \leq l$

$$D_{i,k} D_{i,k'} = D_{i,k'} D_{i,k}$$

$\mathcal{D} = C^*$ -alg generated by $D_{i,k}$, I .

- $D_{i,k} \mathcal{H}'_l \subset \mathcal{H}'_l \leadsto D_{i,k}$ diagonalizable

for $e_1, e_2, \dots$ orthonormal basis for $\mathcal{H}'_1 \subset \mathcal{H}'_2 \subset \dots$

- $(Q_l)_{l=1}^{\infty}$ incr. seq. of fin. rk projs,

$$\lim_{l \rightarrow \infty} D_{i,k} Q_l = Q_l D_{i,k} = 0 \quad \text{for each } i, k$$

$$\leadsto \forall A \in \mathcal{D} \quad \lim_{l \rightarrow \infty} A Q_l = Q_l A = 0$$

$$A_i = \sum_{k=1}^{\infty} 2^{-k} P_{i,k} \quad D_i = \sum_{k=1}^{\infty} 2^{-k} D_{i,k}.$$

$$\leadsto K_i = A_i - D_i = \sum_{k=N+1}^{\infty} 2^{-k} P_{i,k} Q_k \text{ has}$$

norm bdd by $\sum_{k=N+1}^{\infty} 2^{-k} \|P_{i,k} Q_k\| \leq \sum_{k>N} 2^{-k} = 2^{-N} < \varepsilon$
 $\uparrow$ proj. norm 1

Existence of quasicentral unit for $\mathcal{Q} \subset \mathcal{Q}$.

(D. Thm I.9.6 / S. 3.2.15)

Start with an approx. unit $(E_\lambda)_{\lambda \in \Lambda}$ for $\mathcal{Q}$.

we construct a new net $(E_{\lambda'})_{\lambda' \in \Lambda'}$

- $\Lambda' = \{(\lambda, S) \mid \lambda \in \Lambda, S \subset \mathcal{Q} \text{ finite subset}\}$

$\lambda' \leq \mu'$ means $\lambda \leq \mu, S_{\lambda'} \subset S_{\mu'}$.

- $0 \leq E_{\lambda'} \leq I$ s.t. $\begin{cases} E_\lambda \leq E_{\lambda'} \leadsto \text{approx. unit for } \mathcal{Q} \\ \|A E_{\lambda'} - E_{\lambda'} A\| \leq \frac{1}{|S|} \\ \text{for } A \in S \end{cases}$

Fix $A_1, \dots, A_n, \lambda \in \Lambda$.

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix} \in M_n(\mathcal{Q})$$

write $B^{(n)} = \begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix}$ for $B \in \mathcal{Q}$.

$\mathcal{F} = \text{convex hull of } E_\mu \text{ } (\mu \geq \lambda)$

$$\mathcal{J} = \{A F^{(n)} - F^{(n)} A \mid F \in \mathcal{F}\}$$

Claim. $0 \in \mathcal{J}$

Enough to check $\lim_{\mu \rightarrow \infty} f(A E_\mu^{(n)} - E_\mu^{(n)} A) = 0$

for all lin. func. $f: M_n(\mathcal{Q}) \rightarrow \mathbb{C}$

GNS gives $\pi: M_n(\mathcal{Q}) \rightarrow B(\mathcal{H}), x, y \in \mathcal{H}$ s.t.

$$f(B) = (\pi(B)x, y)$$

Then. $\pi(E_\mu^{(n)}) \xrightarrow{\text{soT}} \text{proj. to } \overline{\pi(M_n(\mathcal{Q}))\mathcal{H}} \in \pi(M_n(\mathcal{Q}))'$
 $\pi(M_n(\mathcal{Q}))$ -inv. subsp.

hence $f(A E_\mu^{(n)} - E_\mu^{(n)} A) \rightarrow 0$.

Then we can take $E'_{\lambda} \in \mathcal{F}$ s.t. $\|A E'_{\lambda} - E'_{\lambda} A\| \leq \frac{1}{n}$

we also have $E'_{\lambda} \geq E_{\lambda}$ from $E_{\mu} \geq E_{\lambda} \text{ } (\mu \geq \lambda)$

