

K-theory for C^* -algebras

basic idea: make groups from projections /
invertible elements in a C^* -algebra
up to homotopy

Motivation 1 eigenprojection for Hamiltonian

suppose we have a "space" X ($\mathbb{R}^n$, graph, ...)

positive op. H with discrete spectrum
acting on a function space on X ($L^2(X)$)

$\mathcal{Q}$: C^* -alg of ops on $L^2(X)$ containing H

$\Rightarrow$ eigenprojections $E_\lambda = 1_{\{\lambda\}}(H) \in \mathcal{Q}$

continuous change of $H \rightarrow$ cont. change of
eigen projs

Motivation 2 Fredholm index

$F \in \mathcal{B}(\mathcal{H})$ Fredholm op. (invertible modulo compact ops)

$\Rightarrow \text{Ind } F = \dim \ker F - \dim \ker F^*$

We have - $\text{Ind}(F+K) = \text{Ind } F$

- $\text{Ind } FF' = \text{Ind } F + \text{Ind } F'$

- $\text{Ind } F_t = \text{Ind } F_0$ if $(F_t)_{0 \leq t \leq 1}$ is a

continuous family of Fredholm ops

$\Rightarrow$ "invertible elems of $\mathcal{Q}(\mathcal{H})$ up to homotopy"

should have a group structure,

Ind is a homomorphism from this group to $\mathbb{Z}$

$K_0(\mathcal{Q})$: formal difference of projections

(Davidson p. 100 / Strung 12.1)

Case 1 unital C^* -algebra

Consider matrix algebras $M_n(\mathcal{Q})$

Step 1

Step 2. consider projections $P \in M_n(\mathbb{Q})$

$$P = P^2 = P^*$$

Step 3. identify homotopic projs $(P_t)_{0 \leq t \leq 1}$

if $[0, 1] \rightarrow M_n(\mathbb{Q})$ is a proj-valued cont. func.
 $t \mapsto P_t$

then identify P_0 with P_1

Step 4. identify $P \in M_n(\mathbb{Q})$ with $\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+k}(\mathbb{Q})$

Step 5. define binary op ("addition") by

$$[P] + [Q] = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in M_{n+n'}(\mathbb{Q})$$

if $P \in M_n(\mathbb{Q}), Q \in M_{n'}(\mathbb{Q})$

Step 6. $K_0(\mathbb{Q})$: group of formal differences

$$[P] - [Q]. \quad (\text{so } [P] - [Q] = \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} - \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix})$$

Write $K_0(\mathbb{Q})_+ = \{[P] \mid \text{some proj } P \in K_0(\mathbb{Q})\}$

Prop (D. Low IV.1.4.(b) / S. 12.1.1.(ii) up to 12.1.5)

$K_0(\mathbb{Q})$ is a commutative group

i.e. $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ and $\begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}$ are related by above equivalence.

Proof. Step 1. we may assume $P, Q \in M_n(\mathbb{Q})$

use $P \sim \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, Q \sim \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$ to find a

common size

Step 2. conjugate $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ by $\begin{bmatrix} (\cos \frac{\pi}{2} t) I_n & (\sin \frac{\pi}{2} t) I_n \\ (\sin \frac{\pi}{2} t) I_n & (\cos \frac{\pi}{2} t) I_n \end{bmatrix}$

- unitary conj. give projections.

- at $t=0$ $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ itself.

$$\text{at } t=1 \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P & \\ & Q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} Q & \\ & P \end{bmatrix}$$

Fact Homotopy rel. can be replaced by

$$P = V^*V, Q = VV^* \Rightarrow P \text{ and } Q \text{ equiv.}$$

Obs. $Q \mapsto K_0(Q)$ is functorial:

any hom. $Q \xrightarrow{f} B$ induces hom $K_0(Q) \xrightarrow{f_*} K_0(B)$

Case 2. nonunital Q .

take unitization $\tilde{Q} = Q \oplus \mathbb{C}$.

$$\leadsto \text{homs } \mathbb{C} \xrightarrow{\eta} \tilde{Q} \xrightarrow{\xi} \mathbb{C} \quad \xi\eta = \text{Id}_{\mathbb{C}}$$

$$\leadsto K_0(\mathbb{C}) \xrightarrow{\eta_*} K_0(\tilde{Q}) \xrightarrow{\xi_*} K_0(\mathbb{C}) \quad \xi_*\eta_* = \text{Id}_{K_0(\mathbb{C})}$$

so $K_0(\mathbb{C})$ is a direct summand of $K_0(\tilde{Q})$

set $K_0(Q) = \ker \xi_* = \text{complement of } K_0(\mathbb{C})$

$$\text{Ex. } K_0(\mathbb{C}) \cong K_0(M_n) \cong K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$$

$$P \leftrightarrow \text{rk } P.$$

Ex. $K_0(C(X))$ for compact sp. X .

diff of maps $X \xrightarrow{f} \{\text{Projs in } M_n\}$

- up to homotopy $(f_t)_{0 \leq t \leq 1} \Rightarrow f_0 \sim f_1$

- corner emb. $f \sim \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$

($\equiv$ diff. of vector bundles on X)

$$K_0(C(\mathbb{T})) \cong \mathbb{Z}, \quad K_0(C(S^2)) \cong \mathbb{Z}^2, \dots$$

Ex. $K_0(B(\mathcal{H})) = 0$ for infinite dim $\mathcal{H}$.

homotopy between $P = V^*V$ and $Q = VV^*$

$$U = \begin{bmatrix} V & I - \overbrace{VV^*}^Q \\ \underbrace{V^*V}_P - I & V^* \end{bmatrix} \text{ is unitary, conn. to } \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\text{by } U_t = \begin{bmatrix} \cos \frac{\pi}{2} t V & I - (1 - \sin \frac{\pi}{2} t) VV^* \\ (1 - \sin \frac{\pi}{2} t) V^*V - I & \cos \frac{\pi}{2} t V^* \end{bmatrix}$$

$$U \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} V V^* V & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^* & P-I \\ I-Q & V \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \quad ($$

$(P-I)P=0$