

Recap $K_0(\mathcal{A}) = \{[P] - [Q] \mid P, Q \in \mathcal{M}_n(\mathcal{A})\}$
 homotopy $P \sim \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$, $\overset{\text{proj}}{[P] + [Q]} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$

$K_1(\mathcal{A})$: group of invertible elements up to homotopy.

(Davidson p. 233 / Strung p. 12.4)

We start with unital C^* -algs.

Step 1 take $\mathcal{M}_n(\mathcal{A})$ $n=1, 2, \dots$

Step 2 consider invertible elements $X \in \mathcal{M}_n(\mathcal{A})$

Step 3 identify homotopic invertible elements.

$(X_t)_{0 \leq t \leq 1}$ from cont. map $[0, 1] \rightarrow \mathcal{M}_n(\mathcal{A})$
 taking invertible elements; $X_0 \sim X_1$

Rem. We can restrict to unitary elements

$$X = U |X| \quad (|X| = \sqrt{X^* X} \text{ invertible, } U = X |X|^{-1})$$

$$X_t = U |X|^t \text{ connects } U \text{ to } X$$

Step 4. identify $U \in \mathcal{M}_n(\mathcal{A})$ with $\begin{bmatrix} U & \\ & I_k \end{bmatrix} \in \mathcal{M}_{n+k}(\mathcal{A})$

Step 5 define binary op by

$$[U] + [V] = \begin{bmatrix} U & \\ & V \end{bmatrix} \in \mathcal{M}_{n+m}(\mathcal{A}).$$

Prop. (5.12.4.2) $U, V \in \mathcal{M}_n(\mathcal{A})$ invertible.

$$\begin{bmatrix} U & \\ & V \end{bmatrix} \sim UV \text{ by Steps 3 and 4.}$$

Proof: $R_t = \begin{bmatrix} \cos \frac{\pi}{2} t & -\sin \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t & \cos \frac{\pi}{2} t \end{bmatrix} \rightsquigarrow R_t \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} R_t^*$ connects.

$$\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \text{ with } \begin{bmatrix} I & \\ & V \end{bmatrix} \rightsquigarrow \text{prod. with } \begin{bmatrix} U & \\ & I \end{bmatrix} \text{ gives}$$

$$\begin{bmatrix} U & \\ & V \end{bmatrix} \sim \begin{bmatrix} UV & \\ & I \end{bmatrix} \text{ Step 4 gives } \sim UV.$$

Cor. the semigroup of invertibles up to homotopy

$U \sim \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$ is a commutative group. ()

(inverse is given by $[U^{-1}]$).

Def. we write $K_1(A)$ for this comm. group.

Ex. $K_1(\mathbb{C}) = 0$; $GL_n(\mathbb{C})$ is connected.

$\Rightarrow U \sim V$ for all U, V

Again hom. $f: A \rightarrow B$ of unital C^* -algs give

$K_1(A) \xrightarrow{f_*} K_1(B)$ hom. of comm. grps. ()

When A is not unital, we can look at $K_1(\tilde{A}) \xrightleftharpoons[\eta_*]{\xi_*} K_1(\mathbb{C})$

$\mathbb{C} \xrightarrow{\eta} \tilde{A} \xrightarrow{\xi} \mathbb{C}$ but $K_1(\mathbb{C}) = 0$

$\leadsto$ We can set $K_1(A) = K_1(\tilde{A})$ for nonunital A .

Ex. $K_1(C(\mathbb{T})) \cong \mathbb{Z}$ Given $U \in M_n(C(\mathbb{T}))$

take $z \mapsto \det(U_z) \in \mathbb{C} \setminus \{0\}$ to get

a func. $\mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$, then compute the winding number. ()

Ex. $K_1(K(\mathbb{N})) \rightarrow \mathbb{Z}$, $[F] \mapsto \text{Ind } F$ is well-defined
(in fact an isom.)

Computing K_1 as K_0

Suspension (S. 12.4.6) $SA = \{f: [0, 1] \rightarrow A \mid \text{cont}, f(0) = f(1) = 0\}$

another model $C_0(\mathbb{R}, A)$. $f: \mathbb{R} \rightarrow A$, $\lim_{x \rightarrow \pm\infty} f(x) = 0$

Thm (S. 12.4.7) $K_1(A) \cong K_0(SA)$ ()

Construction of $K_1(\mathcal{Q}) \rightarrow K_0(S\mathcal{Q})$

$U \in M_n(\tilde{\mathcal{Q}})$ invertible we may assume $\varepsilon(U) = I_n$

Take a homotopy $(V_t)_{0 \leq t \leq 1}$ of invertibles in $M_{2n}(\tilde{\mathcal{Q}})$

from I_{2n} to $\begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix}$ s.t. $\varepsilon(V_t) = I_{2n}$

$P_t = V_t \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} V_t^{-1}$ idempot. in $M_{2n}(\tilde{\mathcal{Q}})$, $\varepsilon(P_t) = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$

$P_0 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = P_1 \rightsquigarrow P$ is idempot. in $M_{2n}(S\tilde{\mathcal{Q}})$

$(P - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix})_t = P_t - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = P_t - \varepsilon(P_t) \in \mathcal{Q}$

Take $[P] - [I_n] \in K_0(S\mathcal{Q})$ as the image of $[U]$
well-defined: if $(W_t)_{0 \leq t \leq 1}$ is another homotopy

$X_t = V_t W_t^{-1}$ is a homotop. from I_{2n} to I_{2n}
(path in $M_{2n}(S\tilde{\mathcal{Q}})$)

$Q_t = W_t \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} W_t^{-1} \rightsquigarrow P_t = X_t Q_t X_t^{-1}$

$\rightsquigarrow [P] = [Q]$ by homotopy

$[U] = [U'] \rightsquigarrow$ similar arg. gives $[P] = [P']$

Injective: $U \rightsquigarrow [P] - [I_n]$, $U' \rightsquigarrow [P'] - [I_n]$, $[P] = [P']$

would give $P' = X P X^*$ for some unitary $X \in M_{2n}(S\tilde{\mathcal{Q}})$

Claim: $V'^* X V$ commutes with $\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} V'^* X V \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} &= V'^* X P V = V'^* P' X V \\ &= \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} V'^* X V \end{aligned}$$

so $V'^* X V = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ for $C, D \in M_n(S\tilde{\mathcal{Q}})$
(invertible)

$C_0 = V_0'^* X_0 V_0 = X_0 = I_n$, $C_1 = V_1'^* X_1 V_1 = U'^* U$

i.e. U' and U are homotopic by $(U' C_t)_{0 \leq t \leq 1}$

Surjective take $[P] - [I_n] \in K_0(S\tilde{A})$

so $P : [0, 1] \rightarrow \mathcal{M}_k(S\tilde{A})$, $P_0 = I_n = P_1$ ()

we want $U \in \mathcal{M}_n(\tilde{A})$ inv. $(V_t)_{0 \leq t \leq 1}$ homotop
between I_{2n} and $\begin{bmatrix} U & \\ & U^{-1} \end{bmatrix}$ s.t. $V_t \begin{bmatrix} I_n & \\ & 0 \end{bmatrix} V_t^{-1}$

homotopic to $(P_t)_t$ ($2n \leq k$)

Step 1 find $(W_t)_{0 \leq t \leq 1} \subset \mathcal{M}_k(S\tilde{A})$ invertibles

$$P_t = W_t I_n W_t^* \quad W_0 = I_k$$

consider $X_t = \frac{1}{2}((2P_t - I)(2P_{t_0} - I) + I)$ for $t \geq t_0$

- if P_t is close to P_{t_0} , $X_t \sim X_{t_0} = I$.

so X_t invertible.

$$X_t P_{t_0} = \frac{1}{2}((2P_t - I)P_{t_0} + P_{t_0}) = P_t P_{t_0}$$

$$P_t X_t = \frac{1}{2}(P_t(2P_{t_0} - I) + P_t) = P_t P_{t_0}$$

$$\leadsto P_t = X_t P_{t_0} X_t^{-1}$$

$\leadsto$ collecting X_t on small intervals will give W_t

Step 2 at $t=1$ $P_1 = I_n$ forces $W_1 = \begin{bmatrix} U & \\ & Y \end{bmatrix}$

$Y \in \mathcal{M}_{k-n}(\tilde{A})$ inv.

we may assume $\sim_{\text{homotop}} \begin{bmatrix} U^{-1} & \\ & I_{k-2n} \end{bmatrix}$ ()

consider $\begin{bmatrix} W_t & \\ & W_t^{-1} \end{bmatrix}$ connecting $I_{2k} \sim \begin{bmatrix} U & & \\ & Y & \\ & & U^{-1} \\ & & & Y^{-1} \end{bmatrix}$

then use homotop $\begin{bmatrix} Y & & \\ & U^{-1} & \\ & & Y^{-1} \end{bmatrix} \sim \begin{bmatrix} U^{-1} & & \\ & I & \end{bmatrix}$

Step 3 combining homotop from Steps 1 & 2 we get
the desired homotopy V_t ()