

Recap $K_1(\mathcal{A}) = \{ \text{invertible } U \in M_n(\mathcal{A}) \} / \text{homotopy}$
 $[U] + [V] = [UV] = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \quad U \sim \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$

$$K_1(\mathcal{A}) \cong K_0(S\mathcal{A}) \quad S = \{ f \in C([0,1], \mathcal{A}) \mid f(0) = 0 = f(1) \}$$

Today's goal:

- higher K-groups $K_n(\mathcal{A})$.
 (in fact $K_n(\mathcal{A}) \cong K_0(\mathcal{A})$ or $K_1(\mathcal{A})$ depending on $n \bmod 2$)
- ideal $\mathcal{I} \subset \mathcal{A}$ gives exact seq.

$$\cdots \rightarrow K_{n+1}(\mathcal{A}/\mathcal{I}) \xrightarrow{\partial} K_n(\mathcal{I}) \rightarrow K_n(\mathcal{A}) \rightarrow K_n(\mathcal{A}/\mathcal{I}) \rightarrow \cdots$$

$\uparrow$ new "boundary" map. $\uparrow$ canonical maps

Set $S^n \mathcal{A} = \underbrace{S(-) \circ S(-) \circ \cdots \circ S(-)}_{n \text{ times}}(\mathcal{A}) \cong \{ f: [0,1]^n \rightarrow \mathcal{A} \mid f|_{\partial} = 0 \}$

($[0,1]$ rpt $\rightsquigarrow (\mathcal{A}^{[0,1]})^{[0,1]} \cong \mathcal{A}^{[0,1]^2}$, etc.)

and $K_n(\mathcal{A}) = K_0(S^n \mathcal{A})$; consistent with K_1 .

Ex. $K_n(\mathbb{C}) = K_0(C_0(\mathbb{R}^n)) \cong \ker(K_0(C(S^n)) \xrightarrow{\text{ev}_*} K_0(\mathbb{C}))$

Functoriality
homotop.

How to relate $\mathcal{A}$, $S\mathcal{A}$, $S^2\mathcal{A}$.

"toy model" $\mathcal{A} = \mathbb{C}$

recall: Toeplitz alg. $\mathcal{T} = \{ T_f + K \in \mathcal{B}(H^2(\mathbb{T})) \mid f \in C(\mathbb{T}), K \in \mathcal{K}(H^2(\mathbb{T})) \}$

$\mathcal{K}(H^2(\mathbb{T})) \rightarrow \mathcal{T} \rightarrow C(\mathbb{T})$ gives $C(\mathbb{T}) \cong \mathcal{T}/\mathcal{K}$.

$K \mapsto K, T_f + K \mapsto f$.

$S\mathbb{C} \cong \{ f \in C(\mathbb{T}) \mid f(1) = 0 \}$ $f(e^{2\pi i t})$ in $S\mathbb{C}$

$\mathcal{T}_0 = \{ T_f + K \mid f \in S\mathbb{C} \}$. gives

$\mathcal{K} \rightarrow \mathcal{T}_0 \rightarrow S\mathbb{C}$. short ex. seq.

write $\text{Con } \mathcal{A} = \{ f: [0,1] \rightarrow \mathcal{A} \mid f(0) = 0 \}$

$S\mathcal{A} \rightarrow \text{Con } \mathcal{A} \xrightarrow{\text{ev}_*} \mathcal{A}$

In general, take tensor product

$$\mathcal{K} \otimes \mathcal{A} \rightarrow \mathcal{T}_0 \otimes \mathcal{A} \rightarrow S\mathbb{C} \otimes \mathcal{A} \cong S\mathcal{A}.$$

$\mathcal{Q} \mapsto K_n(\mathcal{Q})$ assigns

- comm. grp $K_n(\mathcal{Q})$ for each C^* -alg $\mathcal{Q}$.
- hom. $f_*: K_n(\mathcal{Q}) \rightarrow K_n(\mathcal{B})$ for each hom. $f: \mathcal{Q} \rightarrow \mathcal{B}$.

- if $f_t: \mathcal{Q} \rightarrow \mathcal{B}$ ($0 \leq t \leq 1$) is a cont. fam. of homs $(f_0)_* = (f_1)_*$ as a map $K_n(\mathcal{Q}) \rightarrow K_n(\mathcal{B})$.

($P \in M_k(\tilde{S}^n \mathcal{Q})$) $\text{proj} \mapsto f_t(P) \in M_k(\tilde{S}^n \mathcal{B})$
(homotopy of projs.)

- $K_n(\mathcal{Q}) \cong K_n(M_k(\mathcal{Q})) \cong K_n(\mathcal{K} \otimes \mathcal{Q})$

Cor. $K_n(\text{Con } \mathcal{Q}) = 0$

$\text{id}_{\text{Con } \mathcal{Q}}$ and 0 can be connected by

$$f_t: \text{Con } \mathcal{Q} \rightarrow \text{Con } \mathcal{Q}, \quad g \mapsto g^t(s) = \begin{cases} g^{(s-t)} & t \leq s \leq 1 \\ g(0) = 0 & s \leq t \end{cases}$$

$$\leadsto \text{id}_{K_n(\text{Con } \mathcal{Q})} = (\text{id}_{\text{Con } \mathcal{Q}})_* = (0)_* = 0$$

Thm (S. 12.5.7) $\mathcal{I} \subset \mathcal{Q}$ ideal, $\mathcal{B} = \mathcal{Q}/\mathcal{I}$.

$$\leadsto \text{ex. seq.} \quad \rightarrow K_{n+1}(\mathcal{B}) \rightarrow K_n(\mathcal{I}) \rightarrow K_n(\mathcal{Q}) \rightarrow K_n(\mathcal{B}) \rightarrow \dots$$

Cor (Bott periodicity) $K_{n+2}(\mathcal{Q}) \cong K_n(\mathcal{Q})$

We want $K_1(S \mathcal{Q}) \cong K_0(\mathcal{Q})$.

apply Thm to $\mathcal{K} \otimes \mathcal{Q} \rightarrow \mathcal{I}_0 \otimes \mathcal{Q} \rightarrow \mathcal{Q}$

and check $K_n(\mathcal{I}_0 \otimes \mathcal{Q}) = 0$

Toeplitz
Calkin Ideas behind Thm

Part 1. $K_0(\mathcal{I}) \xrightarrow{i_*} K_0(\mathcal{Q}) \xrightarrow{p_*} K_0(\mathcal{B})$ exact in the middle
nontrivial part: $p_*([P] - [I_k]) = 0 \Rightarrow [P] - [I_k] = i_*([Q] - [I_k])$ for some

$Q \in M_k(\tilde{\mathcal{Q}})$ proj
- up to $P \sim \begin{bmatrix} P & \\ & I_m \end{bmatrix} \xrightarrow{\exists \text{ inv. } U \in M_k(\tilde{\mathcal{B}})} U p(P) U^{-1} = I_k$ st.
from $p_*([P] - [I_k]) = 0$

- $\begin{bmatrix} U & \\ & U^{-1} \end{bmatrix}$ is in the same conn. comp. as I_{2k}

(cont) can be written as $e^{A_1} \dots e^{A_n}$ $A_i \in M_{2k}(\hat{\mathcal{B}})$

$$\leadsto \exists \text{ lift } W \in M_{2k}(\tilde{\mathcal{Q}}) \text{ for } \begin{bmatrix} U & \\ & U^{-1} \end{bmatrix}$$

$$\leadsto Q = W \begin{bmatrix} P & \\ & 0 \end{bmatrix} W \text{ is an idempot,}$$

$$\phi(Q) = \begin{bmatrix} U & \\ & U^{-1} \end{bmatrix} \begin{bmatrix} \phi(P) & \\ & 0 \end{bmatrix} \begin{bmatrix} U & \\ & U^{-1} \end{bmatrix} = \begin{bmatrix} I_k & \\ & 0 \end{bmatrix}$$

hence belongs to $M_{2k}(\mathcal{Q})$.

Part 2. construction of $K_1(\mathcal{B}) \xrightarrow{\cong} K_0(\mathcal{Q})$

from $U \in M_k(\hat{\mathcal{B}})$ invertible

again find $W \in M_{2k}(\tilde{\mathcal{Q}})$ lifting $\begin{bmatrix} U & \\ & U^{-1} \end{bmatrix}$

$$Q = W \begin{bmatrix} I_k & \\ & 0 \end{bmatrix} W^{-1} \text{ belongs to } M_{2k}(\mathcal{Q})$$

$$\leadsto \gamma([U]) = [Q] - [I_k] \in K_0(\mathcal{Q}) \text{ well-defined}$$

Part 3 interpreting $K_1(\mathcal{B}) \rightarrow K_0(\mathcal{Q}) \rightarrow K_0(\mathcal{Q})$

as $K_0(\mathcal{Q}') \rightarrow K_0(\mathcal{Q}') \rightarrow K_0(\mathcal{B}')$

mapping cone: $\{(A, f) \in \mathcal{Q} \oplus C([0, 1], \mathcal{B}) \mid f(1) = 0, p(1) = f(0)\}$
as $\mathcal{Q}'$

$$\mathcal{Q}' \rightarrow \mathcal{Q} \text{ by } (A, f) \mapsto A$$

$$\text{kernel } \mathcal{Q}' = \{(0, f) \mid f(1) = 0, f(0) = 0\} \\ = \Sigma \mathcal{B}$$

We want $K_0(\mathcal{Q}) \cong K_0(\mathcal{Q}')$. $\mathcal{Q} \rightarrow \mathcal{Q}'$, $A \mapsto (A, 0)$

$\mathcal{Q}' / \mathcal{Q} = \text{Con } \mathcal{B}$ contractible, hence $K_0(\mathcal{Q}) \rightarrow K_0(\mathcal{Q}')$

$$C = \{f \in C([0, 1], \mathcal{Q}) \mid f(1) \in \mathcal{Q}\}$$

homotopic to $\mathcal{Q}$ ($\mathcal{Q} \xrightarrow[\text{out}]{\text{const}} C$)

$$\text{Con } \mathcal{Q} \rightarrow C \rightarrow \mathcal{Q}' \text{ gives } K_0(C) \xrightarrow{\cong} K_0(\mathcal{Q}')$$

$$f \mapsto (f(0), p(f)).$$

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