

Unions / inductive limits of C^* -algs

Interesting C^* -algebras often appear as "limits" of smaller C^* -algebras. $(\mathcal{Q}_n)_{n=1}^\infty$

Ex. $\mathcal{K}(\mathcal{H})$ = compl. of fin rk ops.
 = closure of $\bigcup_{n=1}^\infty \mathcal{B}(\mathcal{H}_n)$ for f.d. $\mathcal{H}_n \subset \mathcal{H}$
 (increasing seq. $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots$)

Ex. $X = \prod_{n=1}^\infty \{0, 1\}$ Cantor space

proj maps $X \xrightarrow{p_k} \prod_{n=1}^k \{0, 1\} = X_k$

$C(X_k) \cong \mathbb{C}^{2^k}$ ($|X_k| = 2^k$, commutative alg)

$C(X_k) \hookrightarrow C(X)$ by $f \mapsto f(p_k(\cdot))$

$C(X)$ = closure of $\bigcup_{k=1}^\infty C(X_k)$

Def. $\mathcal{Q}$ C^* -alg. $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots \subset \mathcal{Q}$ increasing
 seq. of subalgs. If $\forall A \in \mathcal{Q}, \varepsilon > 0 \exists n$
 $B \in \mathcal{Q}_n$ s.t. $\|A - B\| < \varepsilon$ ($\bigcup_{n=1}^\infty \mathcal{Q}_n$ dense in $\mathcal{Q}$)
 we say $\mathcal{Q}$ is the inductive limit $\varinjlim \mathcal{Q}_n$

More abstractly: suppose Λ is a directed set

$(\mathcal{Q}_\lambda)_{\lambda \in \Lambda}$ is a directed system of C^* -algs

i.e. $\lambda \leq \mu \rightsquigarrow f_{\lambda\mu}: \mathcal{Q}_\lambda \rightarrow \mathcal{Q}_\mu$ $*$ -hom.

$f_{\lambda\lambda} = \text{id}$, $f_{\mu\nu} f_{\lambda\mu} = f_{\lambda\nu}$ for $\lambda \leq \mu \leq \nu$

We get a $*$ -alg with C^* -seminorm by

$\mathcal{Q}' = \{ (a, \lambda) \mid \lambda \in \Lambda, a \in \mathcal{Q}_\lambda \} /$
 $(a, \lambda) \sim (f_{\lambda\mu}(a), \mu)$
 for $\lambda \leq \mu$

$\| (a, \lambda) \| = \lim_{\mu \rightarrow \infty} \| f_{\lambda\mu}(a) \|$ ($\leq \|a\|$ by $\|f_{\lambda\mu}\| \leq 1$)

$\mathcal{Q}$ = compl. of $\mathcal{Q}'$ for this seminorm.

(S. 8.2.4 / D. IV. 2).

(cont.) write $Q = \varinjlim Q_\lambda$, the inductive limit of $(Q_\lambda)_\lambda$

Rem. often $f_{\lambda\mu}$ are injective.

$\Rightarrow \|f_{\lambda\mu}(a)\| = a$, hence $\|(a, \lambda)\| = \|a\|$
and we can identify (a, λ) with a

Rem. $f_{\lambda\mu}$ does not have to be unital (even when Q_λ, Q_μ are unital) $B(H_\lambda) \rightarrow B(H_\mu)$

consider $\hat{Q}_\lambda$ (if Q_λ is unital $\hat{Q}_\lambda = Q_\lambda \oplus \mathbb{C}I$)

extend $f_{\lambda\mu}$ to a unital $*$ -hom $\tilde{Q}_\lambda \rightarrow \tilde{Q}_\mu$

$\Rightarrow \varinjlim \tilde{Q}_\lambda \cong \varinjlim Q_\lambda$

Prop. (D. Thm IV 1.3 [S. 12.2.4]) $Q = \varinjlim Q_\lambda$

$\forall P \in Q$ proj, $\varepsilon > 0 \exists \lambda, Q \in Q_\lambda$ proj. $\|P - Q\| < \varepsilon$

Lemma. if $t \in \mathbb{R}$ satisfies $|t - t^2| < \delta < \frac{1}{4}$ then
 $-2\delta < t < \delta$ or $1-2\delta < t < 1+2\delta$

Proof of Prop. find $A = A^* \in Q_\lambda$ $\|P - A\| < \delta_\varepsilon$

so $\|P - A^2\| = \|(P - A) + (P - A)A\| < 3\delta_\varepsilon$

and $\|A - A^2\| \leq \|A - P\| + \|P - A^2\| < 4\delta_\varepsilon$

$\Rightarrow \sigma(A) \subset [-8\delta_\varepsilon, 8\delta_\varepsilon] \cup [1-8\delta_\varepsilon, 1+8\delta_\varepsilon]$

func. calc. gives proj $Q \in A_\lambda$, $\|A - Q\| < 8\delta_\varepsilon$

$\Rightarrow \|P - Q\| < 9\delta_\varepsilon$. $\square$

Thm. (D. IV 3.3 / S. 12.2.4)

$K_0(\varinjlim Q_\lambda) \cong \varinjlim K_0(Q_\lambda)$

Proof. $Q_\mu \rightarrow \varinjlim Q_\lambda$ $A \mapsto (A, \mu)$ gives $K_0(Q_\mu) \rightarrow K_0(\varinjlim Q_\lambda)$

$\Rightarrow$ we get $\varinjlim K_0(Q_\mu) \rightarrow K_0(\varinjlim Q_\lambda)$

this map is surjective by Prop. (for $(\tilde{A}_\lambda)_\lambda$)

Injectivity: we want any homotopy $(P_t)_{0 \leq t \leq 1}$ in $\varinjlim A_\lambda$ already happens in $\exists \lambda \in \Lambda$

i.e. $Q_0, Q_1 \in A_\lambda$, $Q_0 \sim P_0$, $Q_1 \sim Q_1$ (norm.)
and homotop. $(Q_t)_{0 \leq t \leq 1}$

- find $t_0 = 0 < t_1 < \dots < t_N = 1$ s.t. $\|P_{t_i} - P_{t_{i+1}}\| < \frac{1}{8}$.

then λ s.t. $\exists Q_i \sim_{1/8} P_{t_i}$ proj in $M_k(\tilde{A}_\lambda)$

so $\|Q_i - Q_{i+1}\| < \frac{1}{2}$. then $Q_i \sim_{\text{homotop.}} Q_{i+1}$.

so $Q_0 \sim_{\text{homotop.}} Q_1$ in $M_k(\tilde{A}_\lambda)$ $\square$

Ex. $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$,

$\mathcal{K}(\mathcal{H}) = \varinjlim_{\mathcal{H}' \subset \mathcal{H} \text{ f.d.}} B(\mathcal{H}')$, $K_0(\underbrace{B(\mathcal{H}')}_{M_n}) \cong \mathbb{Z}$ ^{rank}

each $B(\mathcal{H}') \rightarrow B(\mathcal{H}'')$ for $\mathcal{H}' \subset \mathcal{H}''$ induce

$$K_0(B(\mathcal{H}')) \rightarrow K_0(B(\mathcal{H}''))$$

Ex. $X = \prod_{n=1}^{\infty} \{0, 1\}$ $K_0(C(X)) \cong C(X, \mathbb{Z})$.

for $X_k = \prod_{n=1}^k \{0, 1\}$. $K_0(C(X_k)) \cong C(X_k, \mathbb{Z})$

use $M_N(C(X_k)) \cong C(X_k, M_N) \cong M_N \oplus \dots \oplus M_N$

to get $K_0(C(X_k)) \cong C(X_k, \mathbb{Z})$.

$$[P] - [Q] \mapsto (x \mapsto \text{rk } P_x - \text{rk } Q_x)$$

$C(X, \mathbb{Z}) = \varinjlim C(X_k, \mathbb{Z})$ locally const. func's

Rem. $K_1(\varinjlim A_\lambda) \cong \varinjlim K_1(A_\lambda)$

consider suspension $S(\varinjlim A_\lambda) \cong \varinjlim S A_\lambda$

