

Finite-dimensional C^* -algebra

(Davidson II.1.1.1 / Strung 8.1.2)

$\mathcal{Q}$ C^* -alg, $\dim \mathcal{Q} < \infty$.

$\leadsto \mathcal{Q} \cong \mathcal{M}_{n_1} \oplus \dots \oplus \mathcal{M}_{n_k}$ for some positive integers $n_1, \dots, n_k$.

Proof outline

Step 1. reduction to $\mathcal{Z}(\mathcal{Q}) \cong \mathbb{C}$
 $\mathcal{Z}(\mathcal{Q})$ fin dim comm. C^* -alg. $\hookrightarrow$ center of $\mathcal{Q}$

$\leadsto$ isom to $C(X)$ X finite set. $\cong \mathbb{C}^n$.

$P_x = \delta_x$ for $x \in X$

$\mathcal{Q} = \bigoplus \underbrace{\mathcal{Q} P_x}_{\text{subalg}} \quad ; \quad A = \sum_{x \in X} A P_x$
 $\text{subalg} \rightarrow A P_x B P_x = A B P_x$

$\mathcal{Z}(\mathcal{Q} P_x) = \mathbb{C} P_x$

Step 2. $\mathcal{Z}(\mathcal{Q}) \cong \mathbb{C} \Rightarrow$ any rep. is faithful

check that $\mathcal{Q}$ is simple. (no nontriv ideal)

Take $0 \neq \mathcal{Q} \subset \mathcal{Q}$, quasicentr. approx unit

$(E_\lambda)_{\lambda \in \mathbb{N}}$: approx unit for $\mathcal{Q}$,
 $A E_\lambda - E_\lambda A \rightarrow 0 \quad (A \in \mathcal{Q})$

By $\dim \mathcal{Q} < \infty$, E_λ converges to some $Z \in \mathcal{Q}$.

(take one faithful rep, then wot-lim .)

Z has to be central, $\mathcal{Z}(\mathcal{Q}) = \mathbb{C} \Rightarrow Z = I$.

Then $A E_\lambda \rightarrow A$ for any $A \in \mathcal{Q}$.

Step 3. irred. rep. π gives $\mathcal{Q} \cong B(\mathcal{H})$

irreducibility : $\pi(\mathcal{Q})' = \mathbb{C} I \Leftrightarrow \pi(\mathcal{Q})'' = B(\mathcal{H})$

double commutant thm $\pi(\mathcal{Q})'' = \text{wot-clos. } \pi(\mathcal{Q})$

$\dim \mathcal{Q} < \infty \Rightarrow$ closure does not give anything new $\square$

AF algebras = inductive limits of fin. dim C^* -algs.
(D. III.2 / S. 8.3.9).

- we usually consider sequences of f.d. C^* -algs.

$$\text{so } \mathcal{Q} = \varinjlim_{n \rightarrow \infty} \mathcal{Q}_n$$

- we may assume $\mathcal{Q}_n \rightarrow \mathcal{Q}_m$ ($n < m$) are injective
(replace $\mathcal{Q}_n$ by its img. in $\varinjlim \mathcal{Q}_n$.)

Examples

$$1. \mathcal{K}(\mathbb{H}) = \varinjlim_{\mathbb{H}' \subset \mathbb{H}, \dim \mathbb{H}' < \infty} \mathcal{B}(\mathbb{H}')$$

$$2. \text{Cantor space } X = \prod_{n=1}^{\infty} \{0, 1\} \hookrightarrow C(X) = \varinjlim C(X_k)$$

$$X_k = \prod_{n=1}^k \{0, 1\}$$

3. canonical anticommutation relation (CAR) algebra

$$\mathcal{Q} = \varinjlim \mathcal{M}_{2^k} \quad \mathcal{M}_{2^k} \rightarrow \mathcal{M}_{2^{k+1}}, \quad A \mapsto \begin{bmatrix} A & \\ & A \end{bmatrix}$$

another way: Hilb. sp $\mathbb{H} \rightarrow \mathcal{Q}_{\mathbb{H}}$

$$\mathcal{Q}_{\mathbb{H}} = C^*\text{-alg} (A_x \text{ loc } \mathbb{H}) \mid A_x A_y^* + A_y^* A_x = \langle y, x \rangle I$$

$$A_x^* \text{ linear in } x, \quad A_x A_y + A_y A_x = 0$$

Intrinsic characterization of AF algebras

(S. Thm 8.3.2, Bratteli)

separable $\hookrightarrow \mathcal{Q}$ is AF if and only if: $\forall A_1, \dots, A_k \in \mathcal{Q}, \varepsilon > 0$

$\exists$ fin. dim. C^* -subalg $\mathcal{B} \subset \mathcal{Q}$, $B_i \in \mathcal{B}$

s.t. $\|A_i - B_i\| < \varepsilon$ for $i=1, \dots, k$.

($\mathcal{Q}$ is locally finite-dimensional.) write $A_i \in \frac{1}{2}\mathcal{B}$

Proof AF $\Rightarrow$ loc fin dim is easier

Loc. fin. dim $\Rightarrow$ AF (take $\tilde{\mathcal{Q}}$ and assume unit).

Fix a dense seq. $(A_n)_{n=1}^{\infty}$ in $\mathcal{Q}_1 = \{A \in \mathcal{Q} \mid \|A\| \leq 1\}$

a decr. seq. $\varepsilon_n \searrow 0$

We will const. a seq. $B_1 \subset B_2 \subset \dots$

of f.d. subalgs $A_i \in_{\varepsilon_n} B_n$ for $i \leq n$

Key Lem (5.8.4.8). $\mathcal{F}, \mathcal{B} \subset \mathcal{Q}$ C^* -subalgs

suppose $N = \dim \mathcal{F} < \infty$, write $\mathcal{F} \cong M_{n_1} \oplus \dots \oplus M_{n_k}$

$\forall \varepsilon > 0 \exists \delta_{\varepsilon, N} > 0$: if $E_{ij}^{(a)} \in_{\delta_{\varepsilon, N}} \mathcal{B}$ for all $E_{ij}^{(a)} \in M_{n_a}$ matrix units

a, i, j then $\exists U \in C^*\text{-alg}(\mathcal{F}, \mathcal{B}) \subset \mathcal{Q}$, $\|U - 1\| \leq \varepsilon$

$$U \mathcal{F} U^* \subset \mathcal{B}$$

Proof of Thm. (induction).

knowing B_n s.t. $A_i \in_{\varepsilon_n} B_n$ for $i \leq n$

take $\mathcal{F} = B_n$ in Key Lem,

and elements $\{E_{ij}^{(a)}, A_1, \dots, A_{n+1}\} = \mathcal{X}$.

by loc. fin. dim we have f.d. $\mathcal{B} \subset \mathcal{Q}$ s.t.

$$E_{ij}^{(a)} \in_{\delta_{\varepsilon_{n+1}/4, N}} \mathcal{B} \quad \text{for } A_i \in_{\varepsilon_{n+1}/2} \mathcal{B}$$

Key Lem gives $U \sim 1_{\varepsilon_{n+1}/2}$, $U B_n U^* \subset \mathcal{B}$.

Then $B_{n+1} = U^* \mathcal{B} U$ is fin. dim.

contains A_i by error $\varepsilon_{n+1}/2 + 2 \cdot \varepsilon_{n+1}/4 = \varepsilon_{n+1}$
($i=1, \dots, n+1$)

Proof of Key Lem (outline)

Step 1: Find projs $F_{ii}^{(a)} \in \mathcal{B}$ $E_{ii}^{(a)} \sim F_{ii}^{(a)}$
(Lem 8.4.4) mutually orthogonal

Step 2: Find partial isometries $F_{ij}^{(a)}$ s.t.
(Lem 8.4.5) $F_{ij}^{(a)} F_{ji}^{(a)} = F_{ii}^{(a)}$, $F_{ji}^{(a)} = F_{ij}^{(a)*}$, $E_{ij}^{(a)} \sim F_{ij}^{(a)}$

Step 3: Use unitaries $U_i^{(a)}$ s.t. $U_i^{(a)} E_{ii}^{(a)} U_i^{(a)*} = F_{ii}^{(a)}$
 $U_i^{(a)} \sim 1$

$$\text{and set } U = \sum_{a,i} F_{i1}^{(a)} U_1^{(a)} E_{1i}^{(a)}$$

This satisfies $U E_{ij}^{(a)} U^* = F_{ij}^{(a)} \in \mathcal{B}$

