

Recap - AF algebras  $\equiv \varinjlim A_\lambda$  for directed system of fin-dim  $C^*$ -algs

- local characterization:  $\overset{\text{SAP}}{Q} \text{ AF} \Leftrightarrow \forall A_1, \dots, A_n \in Q, \varepsilon > 0$   
 $\exists B \subset Q$  f.d. subalg,  $A_i \in_\varepsilon B$

How do we understand an inductive system  $(A_\lambda)_{\lambda \in \Lambda}$ ?

Part 1: Non-unital hom  $M_n \rightarrow M_{n'}$

$$A \mapsto \begin{bmatrix} A & & \\ & \ddots & \\ & & A \\ & & & 0_k \\ \underbrace{\phantom{0_k}}_{m \times m} & & & & \end{bmatrix} \quad \text{for } n' = mn + k \quad \begin{array}{l} \text{standard form} \\ \text{with multiplicity } m \end{array}$$

if  $f: M_n \rightarrow M_{n'}$  is a hom,  $U \in M_{n'}$  is unitary  
 then  $A \mapsto U f(A) U^*$  is also a hom

Lem. These are enough:  $f(A) = U f_{\text{std}, m}(A) U^*$  ( $A \in M_n$ )  
 (D. III. 1.2) for some  $m \leq \frac{n'}{n}$ ,  $U \in M_{n'}$

Sketch: Step 1 look at  $P_i = f(E_{ii})$   $i=1, \dots, n$

these are proj's,  $V_{ij} = f(E_{ij})$   $V_{ij} P_j V_{ij}^* = P_i$

$\Rightarrow \text{rk } P_1 = \text{rk } P_2 = \dots = \text{rk } P_n$  set  $m = \text{rk } P_i$

$$\begin{bmatrix} I_m & & \\ & \ddots & \\ & & 0 \\ \hline & & & 0 \end{bmatrix}_{Q_1}, \begin{bmatrix} 0 & & \\ & \ddots & \\ & & I_m & \\ & & & 0 \end{bmatrix}_{Q_2}, \dots \quad \begin{array}{l} \text{satisfy } P_i \sim Q_i \\ Q_i Q_j = 0 \\ \text{for } i \neq j \end{array}$$

$\Rightarrow \exists$  unitary  $U$  s.t.  $U Q_i U^* = P_i$

Step 2 modify  $U$  to achieve  $U f_{\text{std}, m}(E_{ij}) U^* = f(E_{ij})$

$W_{ij} = U^* f(E_{ij}) U$  is a partial isomet s.t.  $W_{ji} = W_{ij}^*$

$$W_{ij} W_{ij}^* = Q_i, \quad W_{ij}^* W_{ij} = Q_j, \quad W_{ij} W_{jk} = W_{ik},$$

$$\Rightarrow W_{ij} = \begin{bmatrix} 0_m & 0_m & & \\ & \ddots & & \\ & & 0_m & \\ & & & V_{ij} & \dots \\ & & & \vdots & \ddots \end{bmatrix}_i \quad \begin{array}{l} \text{for some unitary } V_{ij} \in M_m \\ V_{ij} V_{jk} = V_{ik}, \quad V_{ji} = V_{ij}^* \end{array}$$

we want a unitary  $U' \in M_{n'}$  s.t.  $U'^* W_{1j} U' = \begin{bmatrix} 0_m & I_m \\ & \vdots \\ & 0_m \end{bmatrix}$  ( $j \geq 2$ )

$$\begin{aligned} (\Rightarrow U'^* W_{ij} U' &= U'^* W_{1i}^* W_{1j} U' = (U'^* W_{1i} U')^* U'^* W_{1j} U' \\ &= \begin{bmatrix} 0_m & & \\ & \ddots & \\ & & I_m \end{bmatrix}_i, \text{ finishing the proof}) \end{aligned}$$

$$U' = \begin{bmatrix} I_m & & \\ & V_{12}^* & \\ & & \ddots \\ & & & V_{1k}^* \end{bmatrix} \text{ will do. } \square$$

Part 2. (Non-unital) hom  $M_{n_1} \oplus \dots \oplus M_{n_k} \xrightarrow{f} M_{n'}$

write  $I_{n_j}^{(j)}$  for  $0 \oplus \dots \oplus I_{n_j} \oplus \dots$

$R_j = f(I_{n_j}^{(j)}) \in M_{n'}$  ( $j=1, \dots, k$ ) mut. orth. proj. in  $M_{n'}$

we can find unitary  $V \in M_{n'}$  s.t.

$$V R_j V^* = \begin{bmatrix} 0_{r_1 + \dots + r_{j-1}} & \\ & I_{r_j} \\ & & 0_{n' - (r_1 + \dots + r_j)} \end{bmatrix}$$

Combined with Part 1:  $f$  is determined by

multiplicities  $m_j$  for each  $M_{n_j}$  ( $\sum_{j=1}^k m_j n_j \leq n'$ )

$$f = U \left( \underbrace{f_{\text{std}, m_1} \oplus f_{\text{std}, m_2} \oplus \dots}_{A_1 \oplus \dots \oplus A_k} \right) U^* \text{ for some } U \in M_{n'} \text{ unitary}$$

$$A_1 \oplus \dots \oplus A_k \mapsto \begin{bmatrix} A_1 & & \\ \underbrace{\quad}_{m_1} & A_2 & \\ & \underbrace{\quad}_{m_2} & \ddots \end{bmatrix}$$

$$f \text{ is unital} \Leftrightarrow \sum_{j=1}^k m_j n_j = n'$$

Part 3 (Non-unital) hom  $M_{n_1} \oplus \dots \oplus M_{n_k} \rightarrow M_{n'_1} \oplus \dots \oplus M_{n'_l}$

(Davidson III.2.1) combine  $f_i: M_{n_1} \oplus \dots \oplus M_{n_k} \rightarrow M_{n'_i}$

given by Part 2  $\Rightarrow f$  is determined by multiplicities

$$(m_{ij}), i=1, \dots, l, j=1, \dots, k$$

Composition  $\mathcal{M}_{n_1} \oplus \dots \oplus \mathcal{M}_{n_k} \rightarrow \mathcal{M}_{n'_1} \oplus \dots \oplus \mathcal{M}_{n'_\ell} \rightarrow \mathcal{M}_{n''_1} \oplus \dots \oplus \mathcal{M}_{n''_p}$   
 $(m_{j'j}^{i'})_{j',j} \quad (m_{j''j'}^{i''})_{j'',j'}$

$\rightarrow$  multiplicity is given by  $(\sum_{j'} m_{j''j'}^{i''} m_{j'j}^{i'})_{j'',j}$

Summary: inductive sys.  $\mathcal{Q}_\lambda$  fin. dim.  
 $\mathcal{Q}_\lambda \rightarrow \mathcal{Q}_\mu$  for  $\lambda \leq \mu$

can be encoded by

-  $(n_1^{(\lambda)}, \dots, n_{k_\lambda}^{(\lambda)})$   $\mathcal{Q}_\lambda \cong \mathcal{M}_{n_1^{(\lambda)}} \oplus \dots \oplus \mathcal{M}_{n_{k_\lambda}^{(\lambda)}}$   
 integers  $n_i^{(\lambda)} \geq 1$ .

-  $(m_{ij}^{\mu\lambda})_{i=1, \dots, k_\mu, j=1, \dots, k_\lambda}$  multiplicities for  $\mathcal{Q}_\lambda \rightarrow \mathcal{Q}_\mu$ .

s.t.  $m^{\nu\mu} m^{\mu\lambda} = m^{\nu\lambda}$  for  $\lambda \leq \mu \leq \nu$ .  
 mat prod

Bratteli diagram for  $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots$

(infinite) graph  $\Gamma$  with:

- vertex set  $V = V_1 \sqcup V_2 \sqcup V_3 \sqcup \dots$  partition  
 $v \in V_\ell$  "vertex at level  $\ell$ "
- edges are between  $v \in V_\ell$  and  $w \in V_{\ell+1}$   
 (possibly mult. edges from  $v$  to  $w$ )
- each vertex  $v$  is decorated by an int.  $n_v \geq 1$ .
- $\sum_{w \in V_\ell} \underbrace{\#(\text{edges from } w \text{ to } v)}_{m_{vw}} n_w \leq n_v$  for  $v \in V_{\ell+1}$ .

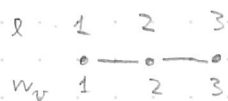
We often ask  $V_1 = \{v_0\}$ ,  $n_{v_0} = 1$

unital case: ask "=" for

Ex.  $\mathcal{K}(\mathcal{Q}^2(\mathbb{N})) = \varinjlim \mathcal{M}_n$

CAR alg.  $= \varinjlim \mathcal{M}_{2^n}$

$C(\{0, 1\}^\omega) = \varinjlim C(\{0, 1\}^k)$



with  $n_v = 1$

