

K-theory for AF algebras

Recall inductive system $(\mathcal{Q}_\lambda)_{\lambda \in \Lambda}$ of C^* -algs.

$$\Rightarrow K_i(\varinjlim_{\lambda} \mathcal{Q}_\lambda) \cong \varinjlim_{\lambda} K_i(\mathcal{Q}_\lambda) \quad (i=0,1)$$

When $\mathcal{Q}_\lambda$ is finite-dimensional $K_1(\mathcal{Q}_\lambda) = 0$.

$$K_0(\mathcal{Q}_\lambda) \cong \mathbb{Z}^{k_\lambda} \quad \text{if} \quad \mathcal{Q}_\lambda \cong M_{n_1} \oplus \dots \oplus M_{n_{k_\lambda}}$$

$$\text{from } K_0(M_n) \cong \mathbb{Z}, \quad K_1(M_n) = 0.$$

Ex. $K_0(\mathcal{K}(\ell^2(\mathbb{N}))) \cong \mathbb{Z}$ from $\mathcal{K} \cong \varinjlim_{n \rightarrow \infty} M_n$

$$M_n \rightarrow M_{n+1}, \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ induces } K_0(M_n) \xrightarrow{\sim} K_0(M_{n+1})$$

$$\text{so } K_0(\mathcal{K}) = \varinjlim_n \mathbb{Z}, \quad \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \text{ as connecting maps}$$

Ex. CAR algebra $\mathcal{Q} = \varinjlim M_{2^k}$ $A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ at each step

$$\text{We have } K_0(M_{2^k}) \cong \mathbb{Z}, \text{ but } K_0(M_{2^k}) \rightarrow K_0(M_{2^{k+1}})$$

$$\text{is not isom; instead } \mathbb{Z} \rightarrow \mathbb{Z}, \quad n \mapsto 2n$$

$$(\text{rk} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} = 2 \text{rk } P)$$

$$K_0(\mathcal{Q}) = \{ \sum (n, k) \mid k=0,1,\dots, n \in \mathbb{Z} \} / \sim$$

"n ∈ ℤ at k-th step" ~(2n, k+1)

$$\text{This is isom to } \mathbb{Z}[\frac{1}{2}] = \{ \frac{n}{2^k} \mid k=0,1,\dots, n \in \mathbb{Z} \}$$

Extra structures on $K_0(\mathcal{Q})$, $\mathcal{Q}$ AF

$$\text{Recall } K_0^+(\mathcal{Q}) = \{ [P] \in K_0(\mathcal{Q}) \mid P \in M_n(\mathcal{Q}) \text{ proj.} \}$$

Thm. [D. IV.2.3-4 / 5, 8.3.14 & 12.1.10]

$$K_0^+(\mathcal{Q}) \subset K_0(\mathcal{Q}) \text{ is a cone: } K_0^+(\mathcal{Q}) \cap (-K_0^+(\mathcal{Q})) = \{0\}$$

$$\text{i.e. } [P] = -[Q] \text{ for proj. } P, Q \in M_n(\mathcal{Q}) \Rightarrow [P] = [Q] = 0$$

Proof $K_0(\mathcal{Q}) = \varinjlim_{\lambda} K_0(\mathcal{Q}_\lambda)$ means $\exists \lambda, P, Q' \in M_n(\mathcal{Q}_\lambda)$



f.d.

$$[P'] = [P] = -[Q] = -[Q']$$

$\mathcal{A}_\lambda \cong M_{n_1} \oplus \dots \oplus M_{n_k} \rightsquigarrow [P']$ is represented by

$$P' \leftrightarrow P'_1 \oplus \dots \oplus P'_k \quad (\text{rk } P'_1, \dots, \text{rk } P'_k) \in \mathbb{N}^k$$

$$\rightsquigarrow [P'] = -[Q'] \quad \text{forces} \quad \text{rk } P'_i = 0 = \text{rk } Q'_i$$

i.e. $P' = 0 = Q'$ $\square$

Scale on $K_0^+(\mathcal{A})$ is given by $\Gamma = \{[P] \mid P \in \mathcal{A}^{\text{proj}}\}$

if $\mathcal{A}$ is unital $[P] \in \Gamma \Leftrightarrow 0 \leq [P] \leq [I]$

$\uparrow$
 $[I] - [P] \in K_0^+$

$$\forall [P] \in K_0^+(\mathcal{A}) \exists n, [Q] \in \Gamma \quad [P] \leq n[Q]$$

(n s.t. $P \in M_n(\mathcal{A})$ would work)

Ex. $\mathcal{A} = \mathbb{K} \quad K_0^+ \cong \mathbb{N} \subset \mathbb{Z} \cong K_0, \quad \Gamma = K_0^+$

$\mathcal{A} = \text{CAR}$ als. $K_0^+ = \left\{ \frac{n}{2^k} \mid n \in \mathbb{N}, k \in \mathbb{N} \right\} \subset \mathbb{Z} \left[\frac{1}{2} \right]$

$$\Gamma = \left\{ \frac{n}{2^k} \mid 0 \leq n \leq 2^k, k \in \mathbb{N} \right\}$$

$\mathcal{A} = C(X), \quad X = \{0, 1\}^\omega \quad K_0^+ = C(X, \mathbb{N}) \subset C(X, \mathbb{Z})$

$$\Gamma = C(X, \{0, 1\})$$

$$(\leftarrow \{K \subset X \mid \text{closed} \& \text{open}\})$$

$\mathcal{A} = M_{n_1} \oplus \dots \oplus M_{n_k} \quad K_0^+ = \mathbb{N}^k \subset K_0 = \mathbb{Z}^k$

$$\Gamma = \{(x_1, \dots, x_k) \mid 0 \leq x_i \leq n_i\}$$

Thm (Elliott; D. IV.4.3 / S. 13.1.8) $\mathcal{A}, \mathcal{B}$ separable AF als

$$\mathcal{A} \cong \mathcal{B} \Leftrightarrow \exists \phi: K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{B}) \text{ s.t. } \phi(\Gamma_{\mathcal{A}}) = \Gamma_{\mathcal{B}}$$

(hence $\phi(K_0^+(\mathcal{A})) = K_0^+(\mathcal{B})$ also)

Proof idea: start with $\mathcal{A} = \varinjlim \mathcal{A}_n, \mathcal{B} = \varinjlim \mathcal{B}_n$

for increasing seqs $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots, \mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$

Up to taking subsequences, we want to find

homomorphisms $A_n \xrightarrow{f_n} B_n$, $B_n \xrightarrow{g_n} A_{n+1}$ s.t.

$$\begin{array}{ccccccc} A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & \dots \\ \downarrow f_1 & \nearrow g_1 & \downarrow f_2 & \nearrow g_2 & \downarrow & \nearrow & \\ B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & \dots \end{array}$$

commutes $\begin{cases} g_n \circ f_n = \text{incl. } A_n \rightarrow A_{n+1} \\ f_{n+1} \circ g_n = \text{incl. } B_n \rightarrow B_{n+1} \end{cases}$

Lem. E : fin dim C^* -alg, $B = \varinjlim B_n$ AF.

$\phi: K_0(E) \rightarrow K_0(B)$ s.t. $\phi(\Gamma E) \subset \Gamma B$

Then $\exists m$, hom $E \xrightarrow{f} B_m$ s.t. $f_* = \phi$ (up to $K_0(B_m) \rightarrow K_0(B)$)

Proof. Step 1. Find m .

Write $E \cong M_{n_1} \oplus \dots \oplus M_{n_k}$

$E^{(i)}$: rk 1 proj. in the i -th summand

Choose m satisfying $\phi([E^{(i)}]) \in K_0(B_m)$ $i=1, \dots, k$

Step 2. Construct $f: E \rightarrow B_m$

Write $B_m \cong M_{n'_1} \oplus \dots \oplus M_{n'_l}$ so $K_0(B_m) \cong \mathbb{Z}^l$

Use $\phi(\Gamma E) \subset \Gamma B \Rightarrow \phi(\Gamma E) \subset \Gamma B_m$ to check

$\mathbb{Z}^k \rightarrow \mathbb{Z}^l$ corresp. to $K_0(E) \xrightarrow{\phi} K_0(B_m)$

is given by $[A_{ij}]_{i=1, \dots, l, j=1, \dots, k}$, $\sum A_{ij} n_j \leq n'_i$

Proof of Th'm Inductively find subseq of

$A_1 \subset A_2, B_1 \subset B_2 \subset \dots, f_n: A_n \rightarrow B_n, g_n: B_n \rightarrow A_{n+1}$

Step 1. Use Lem. for $E = A_1$, find $B_{m_1}, f: A_1 \rightarrow B_{m_1}$

Step 2. Use Lem. for B_{m_1}, A : find $A_{n_2},$

$g_1: B_{m_1} \rightarrow A_{n_2}$

$\exists$ unitary $U \in A_{n_2}$: $g_2 = \text{Ad}_U \circ g_1$ will do

Then $K_0(A_1) \rightarrow K_0(A_{n_2})$ is same as the incl. map