

Tensor products

Lemma

Let A, B be C^* -algebras. Then any C^* -norm on $A \otimes_{\text{alg}} B$ is a cross-norm, that is,

$$\|a \otimes b\|_x = \|a\| \cdot \|b\|$$

Proof

~~usually~~ Assume for simplicity that A and B are unital.

Represent $(A \otimes_{\text{alg}} B, \|\cdot\|)$ isometrically in $B(H)$. It suffices to consider $a, b \geq 0$.

We may assume that $\|a\| = \|b\| = 1$.

Since $\|a \otimes b\|_x \leq \|a \otimes 1\|_x \|1 \otimes b\|_x = \|a\| \cdot \|b\|$, we just have to show that $\|a \otimes b\| \geq 1$.

Let $e = \pi_{[1-\varepsilon, 1]}(a \otimes 1)$, $f = \pi_{[1-\varepsilon, 1]}(1 \otimes b)$.

Then $ef \neq 0$, since otherwise for the function $g : \begin{array}{c} \uparrow \\ 0 \quad \quad 1-\varepsilon \quad 1 \end{array}$ we have $g(a) \neq 0$,

$g(b) \neq 0$, but $g(a) \otimes g(b) = ef (g(a) \otimes g(b)) ef = 0$.

Let $\xi \in efH$, $\|\xi\| = 1$. Then $(a \otimes b)\xi$ is close to ξ (if ε is small), $(1 \otimes b)\xi$ is close to ξ , so $(a \otimes b)\xi$ is close to ξ .

□

Def

A C^* -algebra A is called nuclear if $A \otimes_{\text{alg}} B$ has a unique C^* -norm for every C^* -algebra B .

Proposition

Any AF-algebra is nuclear.

Proof ~~Assume uniqueness~~

$$\text{Let } x = \sum_{i=1}^n a_i \otimes b_i \in A \otimes_{\text{alg}} B.$$

For $\varepsilon > 0$ choose a f.a.l. C^* -subalgebra

$$A_0 \subset A \text{ and } a_i \in A_0 \text{ s.t. } \|a_i - c_i\| < \varepsilon.$$

Then

$$\|x - \sum_i c_i \otimes b_i\|_2 \leq \left(\sum_i \|b_i\| \right) \varepsilon, \text{ so}$$

$$\left\| \sum_i c_i \otimes b_i - \left(\sum_i \|b_i\| \right) \varepsilon \right\| \leq \|x\|_2 \leq \left\| \sum_i c_i \otimes b_i \right\| + \left(\sum_i \|b_i\| \right) \varepsilon$$

Since there is a unique C^* -norm on $A_0 \otimes_{\text{alg}} B$, we see that $\|x\|_2$ is uniquely defined up to $\left(\sum_i \|b_i\| \right) \varepsilon$. Letting $\varepsilon \rightarrow 0$, we get the result. $\square$

Proposition

Any abelian C^* -algebra is nuclear.

Proof

Assume for simplicity that X is compact and B is unital. We have to show that the only C^* -norm on $C(X) \otimes_{\text{alg}} B$ is the one inherited from the supremum-norm on $C(X; B)$.

Let π be an irreducible representation of $C(X) \otimes_{\text{alg}} B$. Then $\pi(C(X)) \in \mathcal{K}$, so π defines a character of $C(X)$, hence a point $t \in X$. Thus π has the form

$$\pi(f \otimes b) = f(t) p(b), \text{ where } p \text{ is irreducible.}$$

Let $\|\cdot\|_2$ be a C^* -norm on $C(X) \otimes_{\text{alg}} B$. It follows that

$$\|a\|_2 = \sup \|p(a(t))\|, \text{ for } a \in C(X) \otimes_{\text{alg}} B \subset C(X; B), \text{ where the supremum is}$$

taken over a set of pairs (t, p) ,
 $t \in X$, p is an irreducible rep. of B .

In particular, $\| \cdot \|_\alpha \leq \| \cdot \|$.

To prove the opposite inequality, note
that for any open set $U \subset X$ we have

$$\bigcap_p \ker p = 0, \text{ where} \quad (*)$$

the intersection is taken over all reps. p of B
s.t. there exists $t \in U$ with $(t, p) \in S$.

Indeed, if there exists $b \in \bigcap_p \ker p$, $b \neq 0$,

then choose $f \neq 0$ s.t. $\text{supp } f \subset U$.

Then $f(t)p(b) = 0$ if $(t, p) \in S$, so $\|f \otimes b\|_\alpha = 0$,
which is a contradiction.

For $a \in C(X) \otimes_{\text{alg}} B$ choose t_0 s.t.

$\|a\| = \|a(t_0)\|$. Let $U \subset X$ be s.t.

$$\|a(t) - a(t_0)\| < \varepsilon \quad \text{for } t \in U.$$

Then By (*) there exists (t, p) s.t.

$(t, p) \in S$ s.t. $t \in U$ and $\|p(a(t_0))\| > \|a(t_0)\| - \varepsilon$.

Then

$$\begin{aligned} \|a\|_\alpha &\geq \|p(a(t))\| > \|p(a(t_0))\| - \varepsilon > \|a(t_0)\| - 2\varepsilon \\ &= \|a\| - 2\varepsilon. \end{aligned}$$

hence $\|a\|_\alpha \geq \|a\|$. □