

Recap. - C^* -tensor norms on $\mathcal{A} \otimes \mathcal{B}$: $\|A \otimes B\|_Y = \|A\| \cdot \|B\|$

- minimal norm for $T \in \mathcal{A} \otimes \mathcal{B}$ $T = \sum A_i \otimes B_i$

$$\|T\|_{\min} = \left\| \sum_{i=1}^n \pi_1(A_i) \otimes \pi_2(B_i) \right\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}$$

- maximal norm for $T \in \mathcal{A} \otimes \mathcal{B}$

$$\|T\|_{\max} = \sup \{ \| \varphi(T) \| \mid \varphi : \mathcal{A} \otimes \mathcal{B} \rightarrow B(\mathcal{H}) \}$$

Independence of $\|T\|_{\min}$ on π_1, π_2

Prop. $\pi_1 : \mathcal{A} \rightarrow B(\mathcal{H}_1)$, $\pi_2 : \mathcal{A} \rightarrow B(\mathcal{H}_2)$

faithful representations

$$\text{Then } \left\| \sum_{i=1}^n \pi_1(A_i) \otimes \pi_2(B_i) \right\| = \left\| \sum_{i=1}^n \rho_1(A_i) \otimes \pi_2(B_i) \right\|$$

Proof. Step 1 Given fin. dim $\mathcal{H}' \subset \mathcal{H}_2$ write $P_{\mathcal{H}'}$

for the orth. proj. to $\mathcal{H}'$

$$C \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightsquigarrow \|C\| = \sup_{\mathcal{H}' \subset \mathcal{H}_2 \text{ f.i.}} \|(I \otimes P_{\mathcal{H}'}) C (I \otimes P_{\mathcal{H}'})\|$$

Step 2. $\|(I \otimes P_{\mathcal{H}'}) (\pi_1 \otimes \pi_2)(T) (I \otimes P_{\mathcal{H}'})\|$ is

indep. of faithful rep. π_1

$$\pi_1(\mathcal{A}) \otimes B(\mathcal{H}') \underset{n = \dim \mathcal{H}'}{\simeq} \pi_1(\mathcal{A}) \otimes \mathcal{M}_n \underset{\pi \text{ faithful}}{\simeq} \mathcal{A} \otimes \mathcal{M}_n$$

contains this elem. The C^* -norm on

$\mathcal{A} \otimes \mathcal{M}_n$ is unique. (that of $\mathcal{M}_n(\mathcal{A})$)

Cor. $\|(\pi_1 \otimes \pi_2)(T)\|$ indep. of faithful π_1, π_2 .

Fact. minimal norm is the smallest among C^* -tensor norms

$$\text{i.e. } \|T\|_Y \geq \|T\|_{\min} \text{ for } T \in \mathcal{A} \otimes \mathcal{B}.$$

For $Y = \max$ this is obvious : $\pi_1 \otimes \pi_2$ is one of valid choices for $\varphi : \mathcal{A} \otimes \mathcal{B} \rightarrow B(\mathcal{H})$

Q. When do we see the difference between $\mathcal{Q} \otimes_{\min} B$ and $\mathcal{Q} \otimes_{\max} B$?

A. B cannot be nuclear (excludes commutative, AF).

Consider a (discrete) group G , $\mathcal{H} = \ell^2(G)$

$\Rightarrow$ two unitary reps $\lambda, \rho: G \rightarrow \mathcal{U}(\mathcal{H})$

$$\lambda_g \delta_h = \delta_{gh} \quad , \quad \rho_g \delta_h = \delta_{hg^{-1}} \quad \lambda_g \rho_h = \rho_h \lambda_g$$

$\Rightarrow \mathcal{Q} = C^*_\lambda(G) = \{ \text{lin. comb. of } \lambda_g \mid g \in G \}$ ^{known case}

$B = C^*_\rho(G) = \{ \sim \rho_g \sim \}$
commute in $B(\mathcal{H})$

$\Rightarrow \mathcal{Q} \otimes_{\max} B \rightarrow B(\mathcal{H})$ is well-defined

What can we say if this makes sense as $\mathcal{Q} \otimes_{\min} B \rightarrow B(\mathcal{H})$?

(*) $\mathcal{Q} \xrightarrow{\Delta} \mathcal{Q} \otimes_{\min} B$, $\lambda_g \mapsto \lambda_g \otimes \rho_g$ is well-defined here
(but not to $\mathcal{Q} \otimes_{\max} B$ in general)

$\lambda \otimes \rho: \mathcal{H} \otimes \mathcal{H} \rightarrow (\mathcal{H} \otimes \mathcal{H})$ as unitary reps

Overall we would get $\mathcal{Q} \xrightarrow{\Delta} \mathcal{Q} \otimes_{\min} B \xrightarrow{\lambda \otimes \rho} B(\mathcal{H})$

$$(\lambda \otimes \rho)(\Delta(\lambda_g)(\delta_h)) = \delta_{ghg^{-1}} \quad , \quad \text{for } h=e.$$

$\Rightarrow \varepsilon(A) = ((\lambda \otimes \rho)(\Delta(A))\delta_e, \delta_e)$ satisfies $\varepsilon(\lambda_g) = 1$.

This can happen if and only if $\mathcal{H}$ contains

a seq. of unit vecs $(x_n)_{n=1}^\infty$ s.t. $\|\lambda_g x_n - x_n\| \rightarrow 0 \quad \forall g$.

$\Rightarrow$ Excludes $SL_n(\mathbb{Z})$ $n \geq 2$, etc

Relation to CAR algebra $\mathcal{Q} = \lim_{k \rightarrow \infty} M_{2^k}$ $A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$

$B(\mathcal{H}) \otimes B(\mathcal{H}') \simeq B(\mathcal{H} \otimes \mathcal{H}')$ for fin. dim $\mathcal{H}, \mathcal{H}'$

$$S \otimes T \leftrightarrow (x \otimes y \mapsto Sx \otimes Ty)$$

becomes $M_n \otimes M_{n'} \simeq M_{nn'}$, and

$A \otimes I_{n'}$ can be interpreted as $\begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix}$

n' times

