

Cuntz algebras (Davidson V.4 / Strung 10.3.1)

$n \geq 2$ integer $\leadsto \mathcal{O}_n$: the unital C^* -algebra with

- generators: $S_1, \dots, S_n$.

- relations: $S_i^* S_i = I$ (S_i is an isometry)

$$S_1 S_1^* + \dots + S_n S_n^* = I$$

Rem By $S_i^* S_i = I$ we already know $P_i = S_i S_i^*$ is a projection. We also want them to be mutually orthogonal & add up to identity.

Ex. V_i on $\ell^2(\mathbb{N})$, $V_i(e_k) = e_{nk+i-1}$ ($k \in \mathbb{N}, i=1, \dots, n$) satisfy the relations for S_i

$\leadsto$ rep. $\mathcal{O}_n \xrightarrow{\pi} \mathcal{B}(\ell^2(\mathbb{N}))$, $\pi(S_i) = V_i$

Prop $S_i^* S_j = 0$ for $i \neq j$

Proof $P_i P_j = 0$ means $S_i S_i^* S_j S_j^* = 0$

sandwich by S_i^* , S_j and use $S_i^* S_i = I$ $\square$

Obs. 1. the linear span of S_i formally satisfy

$$(\alpha_1 S_1 + \dots + \alpha_n S_n)^* (\beta_1 S_1 + \dots + \beta_n S_n) = \sum \overline{\alpha_i} \beta_i = (\beta, \alpha)_{\mathbb{C}}$$

$$\text{i.e. } v^* w = (w, v)_{\mathbb{C}}$$

2. the linear span of $S_{i_1} \dots S_{i_k} S_{j_\ell}^* \dots S_{j_1}^*$ for $k, \ell \in \mathbb{N}$, $1 \leq i_x, j_y \leq n$ is closed under product, involution $\leadsto$ dense subalg of $\mathcal{O}_n$

3. we can think of $S_{i_1} \dots S_{i_k}$ as the tensor prod

$$e_{i_1} \otimes \dots \otimes e_{i_k} \in \mathcal{H} \otimes \dots \otimes \mathcal{H}, \quad \mathcal{H} = \mathbb{C}^n$$

$$S_{j_\ell}^* \dots S_{j_1}^* \text{ as } e_{j_\ell}^* \otimes \dots \otimes e_{j_1}^* \in \mathcal{H}^* \otimes \dots \otimes \mathcal{H}^*$$

$$\text{or } (e_{j_1} \otimes \dots \otimes e_{j_\ell})^* \in (\mathcal{H}^{\otimes \ell})^*$$

$$\text{so } S_{i_1} \dots S_{i_k} S_{j_\ell}^* \dots S_{j_1}^* \in \mathcal{B}(\mathcal{H}^{\otimes \ell}, \mathcal{H}^{\otimes k})$$

We have inductive systems $\mathcal{B}(\mathcal{H}^{\otimes 2}, \mathcal{H}^{\otimes k}) \rightarrow \mathcal{B}(\mathcal{H}^{\otimes k+1}, \mathcal{H}^{\otimes k+1})$

$$T \mapsto T \otimes \text{Id}_1$$

this corresponds to $\sum_{i=1}^n S_{i1} \dots S_{ik} S_x S_x^* S_{j1}^* \dots S_{j1}^* ; S_{j1}^* = S_{i1} - S_{ik} S_{j1}^* - S_{j1}^*$

$$\leadsto \bigoplus_{d \in \mathbb{Z}} \varinjlim_k \mathcal{B}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes d+k}) \subset \mathcal{O}_n \text{ dense}$$

$$A+ d=0 : \varinjlim_k \mathcal{B}(\mathcal{H}^{\otimes k}) = \varinjlim_k \mathcal{M}_n^{\otimes k} \text{ UHF algebra}$$

write $\mathcal{I}^n$ for this subalg.

Theorem (D. V. 4.3 / S. 10.3.9) $\exists \mathbb{E} : \mathcal{O}_n \rightarrow \mathcal{I}^n$ faithful positive

map ($A \geq 0 \Rightarrow \mathbb{E}(A) \geq 0$) such that $\mathbb{E}(A) = A$ for $A \in \mathcal{I}^n$
(conditional expectation to $\mathcal{I}^n$).

Proof. Define an action of $\mathbb{T} = \{z \in \mathbb{C} \mid |z|=1\}$ on $\mathcal{O}_n$ by
 $p_z(S_i) = z S_i \quad (i=1, \dots, n)$. so $p_z(S_i^*) = \bar{z} S_i^*$, etc.

(on $\mathcal{B}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes d+k})$, $p_z(T) = z^d T$)

$$\mathbb{E}(A) = \int_0^1 p_{\exp(2\pi i t)}(A) dt \text{ will do}$$

$$(A \geq 0 \Rightarrow p_z(A) \geq A \Rightarrow \int p_z(A) \geq 0) \quad \square$$

Thm (D. V. 4.6 / S. 10.3.12) $X \in \mathcal{O}_n$ nonzero elem. Then

$$\exists A, B \in \mathcal{O}_n \text{ s.t. } A X B = I.$$

Proof. We prove: X nonzero positive $\Rightarrow \exists Z : Z X Z^*$ close to I

(general case: consider $X^* X$).

Step 1. Find $Y \in \bigoplus_{d=-N}^N \mathcal{B}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes d+k})$ $X \sim Y$, $Y = Y^*$

for big enough N, K .

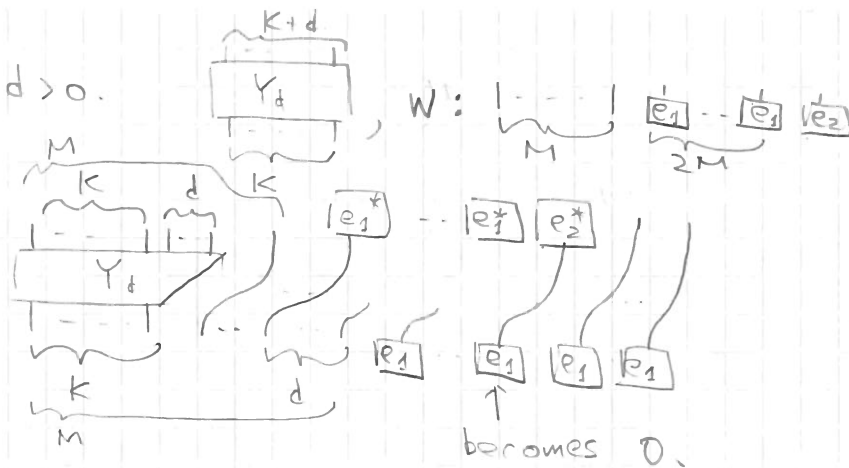
Step 2 $\exists W \in \mathcal{O}_n$ isometry s.t. $W^* Y W = \mathbb{E}(Y)$,
 W commutes with $A \in \mathcal{B}(\mathcal{H}^{\otimes M})$, $M = N+K$.

$$W = \text{Id}_{\mathcal{H}^{\otimes M}} \otimes e_1^{\otimes 2M} \otimes e_2 \in \mathcal{B}(\mathcal{H}^{\otimes M}, \mathcal{H}^{\otimes 3M+1}) \text{ will do.}$$

We need to check $W^* Y_d W = 0$ for $Y_d \in \mathcal{B}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes d+k})$, $d \neq 0$

We may assume $d > 0$.

$W^* Y_d W$ is



Step 3. Take $E \in \mathcal{B}(\mathcal{H}^{\otimes M})$ rk 1 proj, comm. with $\Phi(Y)$

$E \Phi(Y) \geq \frac{1}{\epsilon} \|\Phi(Y)\| E$. (eigenproj. of $\Phi(Y)$)

$v \in \mathcal{H}^{\otimes M}$ unit vec. in the image of E .

$Z = \|\Phi(Y)\|^{-1/2} v^* W^*$ satisfies

$$Z Y Z^* \sim \|\Phi(Y)\|^{-1} v^* W^* Y W E v \\ = \|\Phi(Y)\|^{-1} v^* \Phi(Y) E v \sim I. \quad \square$$

Cor. $\mathcal{O}_n$ is simple. Hence any system of ops $(V_i)_{i=1}^n$ satisfying rels for $(S_i)_{i=1}^n$ generate C^* -alg isom to $\mathcal{O}_n$.

Canonical endomorphism and $K_i(\mathcal{O}_n)$

We have projs $P_i = S_i S_i^* \in \mathcal{O}_n$

$[P_i] = [S_i^* S_i] = [I]$ in $K_0(\mathcal{O}_n)$

$\Rightarrow P_1 + \dots + P_n = I$ gives $n[I] = [I]$ in $K_0(\mathcal{O}_n)$

Fact $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$, $K_1(\mathcal{O}_n) = 0$

In general: $x \in K_i(\mathcal{O} \otimes \mathcal{O}_n) \Rightarrow x = nx$
($(n-1)x = 0$)

Witnessing $(n-1)$ -torsion

$\gamma: \mathcal{O}_n \rightarrow \mathcal{O}_n$, $A \mapsto \sum_{i=1}^n S_i A S_i^*$ endomorphism.

$(T \mapsto \text{Id}_{\mathcal{H}_d} \otimes T, \mathcal{B}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes k+d}) \rightarrow \mathcal{B}(\mathcal{H}^{\otimes k+1}, \mathcal{H}^{\otimes k+1+d}))$

We have $\gamma_i(A) = S_i A S_i^*$ non-unital endom, $\gamma = \sum \gamma_i$

Prop. γ is homotopic to $\text{Id}_{\mathcal{O}_n}$.

Proof. When $U \in \mathcal{O}_n$ is unitary, we get an endomorphism $\phi_U : \mathcal{O}_n \rightarrow \mathcal{O}_n$ by $\phi_U(S_i) = U S_i$.
 Take $U_1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ given by $U_1(x \otimes y) = y \otimes x$

Step 1. $\phi_{U_1} = \gamma$

$$\phi_{U_1}(S_i) = U_1 S_i \quad \text{means} \quad U_1(e_i \otimes \text{Id}_{\mathcal{H}}) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\otimes 2})$$

$$\text{this is } x \mapsto x \otimes e_i = \text{Id}_{\mathcal{H}} \otimes e_i = \gamma(S_i)$$

Step 2. homotopy between $\text{Id}_{\mathcal{O}_n}$ and ϕ_{U_1}

Take homotopy $(U_t)_{0 \leq t \leq 1}$, $U_0 = \text{Id}_{\mathcal{H} \otimes \mathcal{H}}$

$\leadsto \phi_{U_t}$ connects $\text{Id}_{\mathcal{O}_n}$ to ϕ_{U_1} $\square$

Thm $\mathcal{A} : C^*$ -algebra, $x \in K_0(\mathcal{A} \otimes \mathcal{O}_n)$

$$\leadsto nx = x \quad \text{i.e., } (n-1)x = 0 \quad \text{in } K_0(\mathcal{A} \otimes \mathcal{O}_n)$$

Proof. we look at the endomorphism $\text{Id}_{\mathcal{A}} \otimes \gamma$ of $\mathcal{A} \otimes \mathcal{O}_n$

Step 1. $\text{Id}_{\mathcal{A}} \otimes \gamma$ induces trivial (identity) map on K_0

Prop above gives homotopy $(\text{Id} \otimes \gamma)(P) \sim P$ for any proj. $P \in \mathcal{M}_n(\mathcal{A} \otimes \mathcal{O}_n)$.

Step 2. $\text{Id}_{\mathcal{A}} \otimes \gamma$ induces the $n \times$ map on K_0

$$\gamma(A) = \sum_{i=1}^n S_i A S_i^* \quad \text{each } \text{Id}_{\mathcal{A}} \otimes S_i \sim S_i^* \text{ is still an (nonunital) endom. of } \mathcal{A} \otimes \mathcal{O}_n$$

$$Q_i = (I \otimes S_i) P (I \otimes S_i^*) \quad \text{gives the same class as } P.$$

$$(P = V V^*, Q_i = V^* V \text{ for } V = P(I \otimes S_i^*))$$

$$\leadsto [\gamma(P)] = [Q_1 + \dots + Q_n] = \sum [Q_i] = \sum_{i=1}^n [P] = n[P] \quad \square$$