

Group C^* -algebras

We look at locally compact groups

- as topological space: locally compact Hausdorff
- group structures $G \times G \rightarrow G \quad (g, h) \mapsto gh$
 $G \rightarrow G, \quad g \mapsto g^{-1}$
 given by continuous maps

Haar: $\exists$ Borel measure μ_G on G s.t. $\mu_g(E) = \mu_G(E)$
 for any $g \in G, E \subset G$ (Borel) measurable

Note: μ_G is unique up to scaling by pos. numbers

Ex. G discrete $\leadsto$ counting measure

$G = \mathbb{R}^n \leadsto$ Lebesgue measure

$G = \mathbb{T} \leadsto \int f(z) d\mu_{\mathbb{T}}(z) = \int_0^1 f(e^{2\pi i t}) dt$

Modular function: in general we don't have

$$\mu_G(E) = \mu_G(Eg), \text{ but the error is a}$$

$$\text{const. factor: } \mu_G(Eg) = \Delta(g) \mu_G(E)$$

(use up-to-scalar uniqueness of left inv. measures)

$$\leadsto \int_G f(g^{-1}) d\mu_G(g) = \int f(g) \Delta(g)^{-1} d\mu_G(g)$$

Banach $*$ -algebra structure on $L^1(G, \mu_G)$

convolution product:

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) d\mu_G(h)$$

$$\leadsto \|f_1 * f_2\|_{L^1(\mu_G)} \leq \|f_1\|_{L^1(\mu_G)} \|f_2\|_{L^1(\mu_G)}$$

$$\text{i.e. } \int |f_1 * f_2|(g) d\mu_G(g) \leq \int |f_1| d\mu_G \int |f_2| d\mu_G$$

$$\text{use left invariance to get } \int f_2(h^{-1}g) d\mu_G(g) = \int f_2 d\mu_G$$

involution $f^*(g) = \Delta(g)^{-1} \overline{f(g^{-1})}$

$\Rightarrow (f_1 * f_2)^* = f_2^* * f_1^*$

Note. we don't have $\|f^* * f\|_1 = \|f\|_1^2$

Unitary representation of G vs. rep. of $L^1(G, \mu_G)$

$\pi: G \rightarrow \mathcal{U}(\mathcal{H})$ unitary rep

$f \in L^1(G, \mu_G) \mapsto \pi_f = \int f(g) \pi_g d\mu_G(g) \in \mathcal{B}(\mathcal{H})$

i.e. $\pi_f(x) = \int f(g) \pi_g(x) d\mu_G(g)$ for $x \in \mathcal{H}$.

We have $\|\pi_f\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_{L^1(\mu_G)}$

$\pi_{f_1} \pi_{f_2} = \pi_{f_1 * f_2}$, $\pi_f^* = \pi_{f^*}$.

Product: use left invariance of μ_G to get

$$\begin{aligned} \int f_1(g) f_2(h) \pi_g \pi_h d\mu_G(g) d\mu_G(h) \\ = \int f_1(g) f_2(g^{-1}k) \pi_k d\mu_G(g) d\mu_G(k) \end{aligned}$$

Adjoint: $(\pi_f^*(x), y) = (x, \pi_f(y))$

$= \int (x, f(g) \pi_g(y)) d\mu_G(g)$

use $\pi_g^* = \pi_{g^{-1}}$ to get $\overline{f(h^{-1})} \pi_h(x)$.

$\mapsto \Delta(h)^{-1}$ factor from $d\mu_G(h^{-1}) \mapsto d\mu_G(h)$

Non-degenerate rep. $\pi: L^1(G, \mu_G) \rightarrow \mathcal{B}(\mathcal{H})$.

$\hookrightarrow \{\pi_f(x) \mid f \in L^1(G), x \in \mathcal{H}\} \subset \mathcal{H}$ dense span.

write $\mathcal{H}_0$ for this dense span.

$\mapsto$ unitary rep. $G \xrightarrow{\pi} \mathcal{U}(\mathcal{H}_0)$ (extends to $\mathcal{H}$)

by $\pi_g \pi_f(x) = \pi_{f(g^{-1}\cdot)}(x)$.

$$\text{i.e. } (\pi_g \pi_{f_1}(x_1), \pi_{f_2}(x_2)) = (\pi_{f_1}(x), \pi_{g^{-1}} \pi_{f_2}(x_2))$$

Summary

$$\{\text{unitary reps of } G\} \xleftrightarrow{1:1} \{\text{nondeg. reps of } L^1(G, \mu_G)\}$$

C^* -algebra completions of $L^1(G, \mu_G)$.

- the universal group C^* -algebra

$$\|f\|_u = \sup \{ \|\pi f\| \mid \pi \text{ rep. as above} \}$$

$$\leadsto C^*(G) = \text{compl. of } L^1(G, \mu_G) \text{ for } \|\cdot\|_u.$$

- the reduced group C^* -algebra.

$$\|f\|_r = \|\lambda f\|. \quad \text{for the left regular rep.}$$

$$\lambda: G \rightarrow \mathcal{U}(L^2(G, \mu_G)) \quad (\lambda_g f)(h) = f(g^{-1}h)$$

$$(\text{so } \lambda_{f_1}(f_2) = f_1 * f_2 \text{ for } f_2 \in L^2(G, \mu_G))$$

$$\leadsto \{(\text{nondeg.}) \text{ rep of } C^*(G)\} \xleftrightarrow{1:1} \{\text{unitary rep of } G\}$$

representations of $C_r^*(G)$ corresp. to a subclass of unitary reps. of G : $\pi \prec \lambda$ for π on $\mathcal{H}$

$$\forall x \in \mathcal{H}, \varepsilon > 0, K \subset G \text{ cpt. } \exists x_1, \dots, x_n \in L^1(G, \mu_G) \\ t_1, \dots, t_n > 0$$

$$\text{s.t. } (\pi_g(x), x) \sim_\varepsilon \sum t_i (\lambda_g(x_i), x_i) \quad (g \in K).$$

Ex. solvable groups satisfy $C^*(G) = C_r^*(G)$.

(in particular commutative)

$SL_n(\mathbb{R}), SL_n(\mathbb{Z}), IF_n$ ($n \geq 2$) don't satisfy

$$C^*(G) = C_r^*(G) \quad (\text{triv. rep. } \prec \lambda).$$

$$\text{Rem. } C_r^*(G) \subset L(G) = \{p_g \mid g \in G\}'$$

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