

Crossed products (Davidson Chap. VIII; Strung Section 9.4.)

G : locally compact group

$\mathcal{A}$: C^* -algebra

$\alpha_g \in \text{Aut}(\mathcal{A})$ for $g \in G$: action of G on $\mathcal{A}$

($\alpha_g \alpha_h = \alpha_{gh}$; $\forall A \in \mathcal{A}$: $G \rightarrow \mathcal{A}$, $g \mapsto \alpha_g(A)$ cont.)

How should G and $\mathcal{A}$ interact if they are represented on the same Hilbert space?

$\leadsto$ covariant representation $\pi(A), u_g \in \mathcal{B}(\mathcal{H})$ $A \in \mathcal{A}$
 $g \in G$

$$u_g \pi(A) u_g^* = \pi(\alpha_g(A))$$

i.e. adjoint action by $\{u_g\}_{g \in G}$ preserves $\pi(\mathcal{A})$ and implements α

Ex. $\mathbb{R} \curvearrowright C_0(\mathbb{R})$ translation action $(\alpha_t f)(x) = f(x-t)$

$\mathcal{H} = L^2(\mathbb{R})$ with multiplication action of $C_0(\mathbb{R})$

and regular rep. $\{\lambda_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}$

$$\leadsto \lambda_t f \Big|_{C_0(\mathbb{R})} \lambda_t^* g \Big|_{L^2(\mathbb{R})} = \lambda_t f g(t+\cdot) = f(\cdot-t+\cdot) g$$

What is the analogue of $L^1(G, \mu_G)$?

Monally $\{A=f \mid f \in L^1(G, \mu_G), A \in \mathcal{A}\}$

think of f as $\int f(g) u_g d\mu_G(g)$

$$A_1 f_1 A_2 f_2 = \int f_1(s) A_1 u_g f_2(h) A_2 u_h d\mu_G(g) d\mu_G(h)$$

$$= \int f_1(g) f_2(h) A_1 \alpha_g(A_2) u_{gh} d\mu_G(g) d\mu_G(h)$$

$$= \int_{k=gh} \left(\int f_1(g) f_2(g^{-1}k) A_1 \alpha_g(A_2) d\mu_G(g) \right) u_k d\mu_G(k)$$

left invariance of μ_G

$\leadsto$ in terms of the functions $F_i(g) = A_i f_i(g)$

(cont.) $(F_1 * F_2)(k) = \int F_1(g) \alpha_g(F_2(g^{-1}k)) d\mu_G(g)$

represents the product

involution: $(\int A f(g) u_g d\mu_G(g))^* = \int f^*(h) u_h d\mu_G(h) A^*$

for $f^*(h) = \Delta(h)^{-1} \overline{f(h^{-1})}$

use $u_h A^* = \alpha_h(A^*) u_h$ to write this as

$$\int \alpha_h(A^*) f^*(h) u_h d\mu_G(h)$$

$\leadsto f^*(g) = \Delta(g)^{-1} \alpha_g(F(g^{-1})^*)$ represents integrand

More formally, start from $C_c(G; A) = \{F: G \rightarrow A\}$

continuous, compact support $\}$ with $F_1 * F_2, F^*$

and norm $\|F\| = \int \|F(g)\| d\mu_G(g)$

$\leadsto \{ \text{nondeg. rep. of } C_c(G; \mathbb{Q}) \text{ on } H \}$

$$\xleftrightarrow{1:1} \{ \text{cov. rep. of } A, G \text{ on } H \}$$

How to construct covariant representations

For simplicity assume G is discrete, $\alpha: G \curvearrowright \mathbb{Q}$

suppose $\pi_0: \mathbb{Q} \rightarrow \mathcal{B}(\mathcal{H}_0)$ is a representation

Take $\mathcal{H} = \bigoplus_G \mathcal{H}_0$ direct sum of copies indexed by $g \in G$

$\pi: \mathbb{Q} \rightarrow \mathcal{B}(\mathcal{H})$ diagonal on each summand,

at the g -th summand $A \in \mathbb{Q}$ acts by $\pi_0(\alpha_{g^{-1}}(A))$

$u: G \rightarrow \mathcal{U}(\mathcal{H})$ shift index of direct sum

$$u_g(x \text{ at } h\text{-th summand}) = x \text{ at } gh\text{-th summand}$$

$$x \in \mathcal{H}_0 \quad h \in G$$

Prop. The above satisfy $u_g \pi(A) u_g^* = \pi(\alpha_g(A))$

Proof: ^{Step 1} $u_g \pi(A) u_{g^{-1}} (x \text{ at } h\text{-th summand})$

$$= u_g \pi(A) (x \text{ at } g^{-1}h\text{-th summand})$$

$$= u_g (\pi_0(\alpha_{h^{-1}g}(A)) x \text{ at } g^{-1}h\text{-th summand})$$

$$= \pi_0(\alpha_{h^{-1}g}(A)) x \text{ at } h\text{-th summand}$$

Step 2: $\pi(\alpha_g(A)) (x \text{ at } h\text{-th summand})$

$$= \pi(\alpha_{h^{-1}}(\alpha_g(A))) x \text{ at } h\text{-th summand} \quad \square$$

C^* -algebra completion of $C_c(G, \mathbb{Q})$ for

all (non-deg.) reps on Hilbert spaces.

$\leadsto \mathbb{Q} \rtimes_\alpha G$ universal / full crossed product

C^* -alg. compl. for $\bigoplus_G \mathcal{H}_0$, $\pi_0: \mathbb{Q} \rightarrow B(\mathcal{H}_0)$ faithful rep.

$\leadsto \mathbb{Q} \rtimes_{\alpha, r} G$ reduced crossed product.

Non-discrete case:

replace $\bigoplus_G \mathcal{H}_0$ by $L^2(G, \mu_G) \otimes \mathcal{H}_0 \simeq L^2(G; \mathcal{H}_0)$

$$(\pi(A)x)(g) = \pi_0(\alpha_{g^{-1}}(A))x(g)$$

Ex. $\mathbb{C} \rtimes G = C^*(G)$, $\mathbb{C} \rtimes_r G = C_r^*(G)$

$$C_0(\mathbb{R}) \rtimes \mathbb{R} \simeq C_0(\mathbb{R}) \rtimes_r \mathbb{R} \simeq \mathcal{K}(L^2(\mathbb{R}))$$

then covariant rep. on $L^2(\mathbb{R})$ gives
a faithful rep. of $C_0(\mathbb{R}) \rtimes \mathbb{R}$.

