

Examples of crossed products by $\mathbb{Z}$.

action of the group $\mathbb{Z}$ on $\mathcal{Q}$

$\equiv$ giving a single automorphism $\alpha = \alpha_1$ of $\mathcal{Q}$

1. Noncommutative torus / irrational rotation algebra

$\mathcal{Q} = C(\mathbb{T})$. α : rotation of $\mathbb{T}$.

$\alpha(f)(z) = f(e^{-2\pi i \theta} z)$ for some fixed θ

write V for the unitary representing $1 \in \mathbb{Z}$

$$\text{so } V f V^* = \alpha(f)$$

write U for $z \in C(\mathbb{T}) \mapsto V U V^* = e^{-2\pi i \theta} U$

$$\text{i.e. } U V = e^{2\pi i \theta} V U$$

U generates $C(\mathbb{T})$ as a C^* -alg

$\mathcal{Q}_\theta = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is generated by $C(\mathbb{T})$, V^n

$\leadsto C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is generated by U, V .

Fact when θ is an irrational number

- $\mathcal{Q}_\theta$ is a simple C^* -alg.

- $(\mathcal{Q}_\theta)_{sa} = \{x \in \mathcal{Q}_\theta \mid x = x^*\} = \{x \mid x = x^*, \text{ invertible}\}$
dense

Prop - $\mathcal{Q}_\theta$ has a unique state τ s.t.

$$\tau(XY) = \tau(YX) \quad (X, Y \in \mathcal{Q}_\theta)$$

tracial state

$$\tau(U^m V^n) = \begin{cases} 1 & m=n=0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch for the uniqueness of tracial state

$$\text{Step 1. } U U^m V^n U^* = e^{2\pi i n \theta} U^m V^n$$

$$U V^n U^* = (U V U^*)^n = e^{2\pi i n \theta} (U V U^*)^n$$

$$\text{Step 2. } \tau \text{ tracial, } n \neq 0 \Rightarrow \tau(U^m V^n) = 0$$

(Cont.) $\tau(U \times U^*) = \tau(X U^* U) = \tau(X)$

if $X = U^m V^n$ we get $\underbrace{e^{2\pi i n \theta}}_{\neq 1 \text{ by } \theta \text{ irrat.}} \tau(X) = \tau(X)$ (

Step 3 $\tau(U^m V^n) = 0$ if $m \neq 0$

switch the role of U and V $\square$

2. Cuntz algebra.

recall $\Theta_n =$ generators $S_1, \dots, S_n$

relations $S_i^* S_i = I = \sum_{i=1}^n S_i S_i^*$

$\Theta_n \cong \bigoplus_{d \in \mathbb{Z}} \varinjlim_k \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes d+k}), T \mapsto T \otimes \text{Id}_{\mathbb{C}^n}$ (

$S_i \leftrightarrow e_i \in \mathbb{C}^n \cong \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$

Consider another inductive sys. $\mathcal{Q}_k = \mathcal{B}((\mathbb{C}^n)^{\otimes 2k})$ $k=0,1,\dots$

$\mathcal{Q}_k \rightarrow \mathcal{Q}_{k+1}, T \mapsto P \otimes T \otimes \text{Id}_{\mathbb{C}^n}$

with $P =$ orth. prj to the span of e_1 .

Prop. $\mathcal{Q} = \varinjlim \mathcal{Q}_k$ is isom. to $\mathcal{K}(\mathcal{H}) \otimes \mathcal{F}$

with $\mathcal{F} = \varinjlim (\mathcal{B}((\mathbb{C}^n)^{\otimes k}), T \mapsto T \otimes \text{Id}_{\mathbb{C}^n}) \underset{"d=0"}{\subset} \Theta_n$ (

Idea. $\mathcal{H} = \varinjlim (\mathbb{C}^n)^{\otimes k}, x \mapsto e_1 \otimes x$.

$= \underbrace{\left\{ \dots e_1 \otimes e_1 \otimes y \mid y \in (\mathbb{C}^n)^{\otimes k}, k=0,1,2,\dots \right\}}_{\text{completion of}} \quad \square$

Automorphism α of $\mathcal{Q}$!

when $T \in \mathcal{Q}_k$, set $\alpha(T) = P \otimes P \otimes T \in \mathcal{Q}_{k+1}$.

Rem. $\mathcal{Q} =$ image of $1 \in \mathbb{C} = \mathcal{Q}_0$ in $\mathcal{Q}$

$= \underbrace{P \otimes P \otimes \text{Id}_{\mathbb{C}^n} \otimes \text{Id}_{\mathbb{C}^n} \otimes \dots}_{\mathcal{Q}_1 \quad \mathcal{Q}_2 \quad \dots}$

$\leadsto \alpha$ induces an endomorphism of $\mathcal{Q} \mathcal{Q} \mathcal{Q} \cong \mathcal{F}$

$$\alpha \leftrightarrow \alpha' \text{ on } \mathcal{F}, \quad \alpha'(T) = P \otimes T = S_1 T S_1^*$$

Prop. $Q(\mathbb{Q} \rtimes_{\alpha} \mathbb{Z})Q \cong \mathcal{O}_n$.

Sketch. Write U_n for the unitary corresp. to $1 \in \mathbb{Z}$
 $\mathbb{Q} \rtimes_{\alpha} \mathbb{Z}$

Projs. $P \otimes P_i \in \mathcal{Q}_1 = \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$

P_i orth. proj to the span of e_i

Set $S'_i = (P \otimes E_{ii}) U \in \mathbb{Q} \rtimes_{\alpha} \mathbb{Z}$.

Step 1. $S'_i \in Q(\mathbb{Q} \rtimes \mathbb{Z})Q$ i.e. $S'_i = Q S'_i = S'_i Q$

$$S'_i = Q S'_i \text{ from } Q(P \otimes E_{ii}) = P \otimes E_{ii}$$

$$S'_i = S'_i Q \text{ from } U Q U^* = \alpha(Q)$$

$$\text{and } (P \otimes E_{ii}) \alpha(Q) = P \otimes E_{ii} \quad \underbrace{P \otimes P \otimes P \otimes \text{Id}_{\mathbb{C}^n}}_{a_1 a_2 \dots}$$

Step 2. $S'^*_i S'_i = Q$ (unit of $Q(\mathbb{Q} \rtimes \mathbb{Z})Q$)

$$\text{we have } U^*(P \otimes P)U = \alpha^{-1}(P \otimes P) = Q.$$

Step 3. $\sum_{i=1}^n S'_i S'^*_i = Q$

$$\text{we have } \sum_{i=1}^n P \otimes P_i = P \otimes \text{Id}_{\mathbb{C}^n} \in \mathcal{Q}_1.$$

this is Q (img of $1 \in \mathcal{Q}_0$) $\square$

Thm $\mathbb{Q} \rtimes_{\alpha} \mathbb{Z} \cong \mathcal{K}(\mathcal{H}) \otimes \mathcal{O}_n$

Proof. Write $\mathcal{H}_k = (\mathbb{C}^n)^{\otimes k} \subset \mathcal{H}$.

($\mathcal{H}_k \subset \mathcal{H}_{k+1}$ by $x \mapsto e_1 \otimes x$)

Q_k : orth. proj. to $\mathcal{H}_k$. $\Rightarrow Q_k \otimes I_{\mathcal{F}} \in \mathcal{K}(\mathcal{H}) \otimes \mathcal{F} = \mathcal{Q}$.

$Q_k(\mathbb{Q} \rtimes \mathbb{Z})Q_k \cong \mathcal{B}(\mathcal{H}_k) \otimes \mathcal{O}_n$ by Prop.

increasing k means $\mathcal{B}(\mathcal{H}_k) \otimes \mathcal{O}_n \hookrightarrow \mathcal{B}(\mathcal{H}_{k+1}) \otimes \mathcal{O}_n$

$$T \mapsto P \otimes T.$$

$\Rightarrow$ limit is $\mathcal{K}(\mathcal{H}) \otimes \mathcal{O}_n$