

Pimsner - Voiculescu sequence for crossed product by $\mathbb{Z}$,

α : automorphism of C^* -alg $\mathcal{Q}$

$\hookrightarrow$ action of $\mathbb{Z}$ by α^k $k \in \mathbb{Z}$
 $\rightarrow$ crossed prod. C^* -alg $\mathcal{Q} \rtimes_{\alpha} \mathbb{Z}$
 $\hookrightarrow$ induced map $\alpha_* : K_i(\mathcal{Q}) \rightarrow K_i(\mathcal{Q})$

Thm. There is an exact sequence of the form

$$\cdots \rightarrow K_0(\mathcal{Q}) \xrightarrow{\text{Id} - \alpha_*} K_0(\mathcal{Q}) \rightarrow K_0(\mathcal{Q} \rtimes_{\alpha} \mathbb{Z}) \rightarrow K_1(\mathcal{Q}) \xrightarrow{\text{Id} - \alpha_*} K_1(\mathcal{Q}) \rightarrow \cdots$$

and another one with index flip

Examples

1. Noncomm. torus. $\mathcal{Q} = C(\mathbb{T})$, $\alpha(f)(z) = f(e^{-2\pi i \theta} z)$

$\alpha_* : K_i(C(\mathbb{T})) \rightarrow K_i(C(\mathbb{T}))$ is identity map.

(use θ as a homotopy parameter)

$\leadsto$ The exact seq. becomes

$$\cdots \rightarrow K_0(C(\mathbb{T})) \xrightarrow{0} K_0(C(\mathbb{T})) \xrightarrow{1} K_0(\mathcal{Q}_{\theta}) \rightarrow K_1(C(\mathbb{T})) \xrightarrow{0} K_1(C(\mathbb{T})) \rightarrow \cdots$$

$\parallel$ $\mathbb{Z}$ $\parallel$ $\mathbb{Z}$ $\parallel$ $\mathbb{Z}$

$$\leadsto K_0(\mathcal{Q}_{\theta}) \cong \mathbb{Z} \oplus \mathbb{Z} \cong K_1(\mathcal{Q}_{\theta}).$$

2. Cuntz alg. $\mathcal{O}_n$. $K(\mathcal{H}) \otimes \mathcal{O}_n \cong (K(\mathcal{H}) \otimes \mathcal{F}) \rtimes_{\alpha} \mathbb{Z}$

$$\mathcal{F} = \varinjlim M_{n^k}, \quad T \mapsto T \otimes I_n.$$

α induces the endomorphism $T \mapsto P \otimes T$
 $P \in M_n$ $\text{rk } 1$ proj

$$K_0(\mathcal{F}) = \mathbb{Z}[\frac{1}{n}], \quad \alpha_* \text{ corresponds to } x \mapsto \frac{x}{n}.$$

$$K_1(\mathcal{F}) = 0.$$

$$\leadsto 0 \rightarrow K_1(\mathcal{O}_n) \rightarrow \mathbb{Z}[\frac{1}{n}] \xrightarrow{(1-\frac{1}{n})_*} \mathbb{Z}[\frac{1}{n}] \rightarrow K_0(\mathcal{O}_n) \rightarrow 0$$

$\uparrow$ $K_1(\mathcal{F})$ $\uparrow$ $K_1(\mathcal{F})$

$$\leadsto K_1(\mathcal{O}_n) = 0 \quad \text{by } (1 - \frac{1}{n})_* \text{ injective on } \mathbb{Z}[\frac{1}{n}]$$

Prop. $\text{Cok} \left(\mathbb{Z}[\frac{1}{n}] \rightarrow \mathbb{Z}[\frac{1}{n}], x \mapsto (1 - \frac{1}{n})x \right) \cong \mathbb{Z}/(n-1)\mathbb{Z}$
 (hence $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$).

Proof. $(1 - \frac{1}{n})x = (n-1) \frac{x}{n}$.

$\{ \frac{x}{n} \mid x \in \mathbb{Z}[\frac{1}{n}] \} = \mathbb{Z}[\frac{1}{n}]$, hence we need

to compute $R / \underbrace{(n-1)R}_{\text{ideal of } R}$ for $R = \mathbb{Z}[\frac{1}{n}]$.

In the quot. ring we have $n-1 = 0$, i.e. $n=1$.

So $\frac{x}{n^k} = x$ in $R/(n-1)R$.

$\mathbb{Z} \cap (n-1)R = (n-1)\mathbb{Z}$ from n & $n-1$ coprime.

$\leadsto$ the surjective map $\mathbb{Z} \rightarrow R/(n-1)R$, $x \mapsto x$
 has kernel $(n-1)\mathbb{Z}$, hence $\mathbb{Z}/(n-1)\mathbb{Z} \cong R/(n-1)R$

Idea for the Pimsner-Voiculescu theorem.

Find a C^* -alg $\mathcal{I}_{\mathcal{Q}, \alpha}$ (analogue of Toeplitz alg)

- $\mathcal{Q} \otimes \mathcal{K}(\ell^2(\mathbb{N})) \subset \mathcal{I}_{\mathcal{Q}, \alpha}$ as an ideal

- $\mathcal{I}_{\mathcal{Q}, \alpha} / (\mathcal{Q} \otimes \mathcal{K}) \cong \mathcal{Q} \rtimes_{\alpha} \mathbb{Z}$

$\left(\begin{array}{ccccccc} \rightarrow \text{exact seq.} & K_0(\mathcal{Q}) & = & K_0(\mathcal{K} \otimes \mathcal{Q}) & \rightarrow & K_0(\mathcal{I}_{\mathcal{Q}, \alpha}) & \rightarrow & K_0(\mathcal{Q} \rtimes_{\alpha} \mathbb{Z}) \\ & \uparrow & & & & & & \downarrow \\ & & & & & & & K_1(\mathcal{Q}) \end{array} \right)$

- $K_i(\mathcal{I}_{\mathcal{Q}, \alpha}) \cong K_i(\mathcal{Q})$, and $K_i(\mathcal{Q}) \rightarrow K_i(\mathcal{I}_{\mathcal{Q}, \alpha})$

becomes $K_i(\mathcal{Q}) \xrightarrow{\text{id} - \alpha_*^{-1}} K_i(\mathcal{Q})$ (then compose with α_* to get $\alpha_* \circ \text{id}$)

Construction of $\mathcal{I}_{\mathcal{Q}, \alpha}$.

$\mathcal{I} \subset \mathcal{B}(\ell^2(\mathbb{N}))$ Toeplitz alg. generated by
 isometry $S: e_i \mapsto e_{i+1}$. ($\leftrightarrow$ mult. by z on $H^2(\mathbb{T})$)

$\mathcal{I}_{\mathcal{Q}, \alpha} \subset (\mathcal{Q} \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{I}$ generated by $\mathcal{Q}$, $U \otimes S$

Inclusion $\mathcal{Q} \otimes \mathcal{K} = \mathcal{T}_{\mathcal{Q}, \alpha}$

$V = U \otimes S$ satisfies $V^* V - V V^* = I \otimes P_0$

P_0 orth. proj to $\mathcal{Q}_0$ ($S^* S - S S^*$)

$V (I \otimes P_0) V^* = I \otimes P_1$, etc. give $I \otimes P_k \in \mathcal{T}_{\mathcal{Q}, \alpha}$

$(A \otimes P_k) V^j \leftrightarrow \alpha^{-k}(A) \otimes E_{k, k-j} \in \mathcal{Q} \otimes \mathcal{K}$

$\mathcal{T}_{\mathcal{Q}, \alpha}$ for $0 \leq j \leq k$

defines inclusion.

$\mathcal{T}_{\mathcal{Q}, \alpha} / (\mathcal{Q} \otimes \mathcal{K}) \cong \mathcal{Q} \rtimes_{\alpha} \mathbb{Z} \quad r, C(1)$

quotient is a subalg of $\mathcal{A} \rtimes \mathbb{Z} \otimes (\mathcal{T}/\mathcal{K})$

generated by $\mathcal{Q}$, $U \otimes Z$.

$\mathcal{Q} \rightarrow \mathcal{T}_{\mathcal{Q}, \alpha}$, $A \mapsto A \otimes I$ induces $K_i(\mathcal{Q}) \cong K_i(\mathcal{T}_{\mathcal{Q}, \alpha})$

"formal difference." $\text{Id}_{\mathcal{T}_{\mathcal{Q}, \alpha}} - \phi : \mathcal{T}_{\mathcal{Q}, \alpha} \rightarrow \mathcal{T}_{\mathcal{Q}, \alpha}$

$\phi(B) = (I \otimes S) B (I \otimes S^*)$ endom. of $\mathcal{T}_{\mathcal{Q}, \alpha}$

we have $B - \phi(B) \in \mathcal{Q} \otimes \mathcal{K}$.

and get a well-defined map $K_i(\mathcal{T}_{\mathcal{Q}, \alpha}) \rightarrow K_i(\mathcal{Q})$

$\mathcal{Q} \otimes \mathcal{K} \hookrightarrow \mathcal{T}_{\mathcal{Q}, \alpha}$ induces $K_i(\mathcal{Q}) \xrightarrow{\text{Id} - \alpha_*^{-1}} K_i(\mathcal{Q})$

$i=0$, consider proj $Q \in \mathcal{Q}$.

$Q \otimes P \in \mathcal{Q} \otimes \mathcal{K}$ represents $[Q]$
 $\uparrow$ rk 1 proj

while $Q \otimes P = Q \otimes I - Q \otimes S S^*$

in $\mathcal{T}_{\mathcal{Q}, \alpha}$ $Q \otimes S S^* = Q V V^* Q$

$[Q V V^* Q] = [V^* Q Q V] = [\alpha^{-1}(Q)]$