

MAT4460 - Fall 2026 - Solutions for week 36

Exercise 5

Let X denote a locally compact Hausdorff space which is not compact, and let $\tilde{X} = X \cup \{\infty\}$ denote the one-point compactification of X .

Then $\widetilde{C_0(X)}$ is (isometrically) $*$ -isomorphic to $C(\tilde{X})$.

Solution: Set $\mathcal{J} := \{g \in C(\tilde{X}) \mid g(\infty) = 0\}$. Since $\tilde{X}$ is a compact Hausdorff space and $X = \tilde{X} \setminus \{\infty\}$, we get from Exercise 2 b) that the map $f \mapsto \tilde{f}$ defined in this exercise is a $*$ -isomorphism from $C_0(X)$ onto $\mathcal{J}$.

It is then straightforward to check that the map $\phi: \widetilde{C_0(X)} \rightarrow C(\tilde{X})$ defined by

$$\phi((f, \lambda)) = \tilde{f} + \lambda 1_{\tilde{X}}$$

is a $*$ -homomorphism. We note that ϕ is injective:

Assume that $\phi((f, \lambda)) = 0$, i.e., $\tilde{f} = -\lambda 1_{\tilde{X}}$. If $\lambda \neq 0$ we get that $1_{\tilde{X}} = -\frac{1}{\lambda} \tilde{f}$. This implies that $1_X = -\frac{1}{\lambda} f \in C_0(X)$, which is not possible since X is not compact. Hence we must have $\lambda = 0$. This implies that $\tilde{f} = 0$, which gives that $f = 0$, hence that $(f, \lambda) = (0, 0)$.

Next, we note that ϕ is surjective:

Let $h \in C(\tilde{X})$. Set $\lambda := h(\infty) \in \mathbb{C}$ and define $g \in \mathcal{J}$ by $g = h - \lambda 1_{\tilde{X}}$. Then $g = \tilde{f}$ for a (unique) $f \in C_0(X)$, so we get that

$$\phi((f, \lambda)) = \tilde{f} + \lambda 1_{\tilde{X}} = g + \lambda 1_{\tilde{X}} = h.$$

Thus we get that ϕ is a $*$ -isomorphism, which implies that ϕ is isometric (as shown in class).

Exercise 6

Let H be an infinite dimensional Hilbert space.

Then $\mathcal{K}(H) + \mathbb{C} \cdot I_H$ is a C^* -subalgebra of $\mathcal{B}(H)$ and $\widetilde{\mathcal{K}(H)}$ is (isometrically) $*$ -isomorphic to $\mathcal{K}(H) + \mathbb{C} \cdot I_H$:

Solution: One checks readily that $\mathcal{K}(H) + \mathbb{C} \cdot I_H$ is a $*$ -subalgebra of $\mathcal{B}(H)$. Since $\mathcal{K}(H)$ is closed in $\mathcal{B}(H)$ and $\mathbb{C} \cdot I_H$ is finite-dimensional, we get from Exercise 4 that $\mathcal{K}(H) + \mathbb{C} \cdot I_H$ is closed in $\mathcal{B}(H)$. This shows the first assertion.

Next, define $\phi: \widetilde{\mathcal{K}(H)} \rightarrow \mathcal{K}(H) + \mathbb{C}I_H$ by

$$\phi((K, \lambda)) = K + \lambda I_H$$

for all $K \in \mathcal{K}(H)$ and $\lambda \in \mathbb{C}$. It is easy to check that ϕ is a $*$ -homomorphism, which is clearly surjective.

Moreover, the kernel of ϕ is trivial. Indeed, assume that $\phi((K, \lambda)) = 0$. Then $K + \lambda I_H = 0$, so $K = -\lambda I_H$. If $\lambda \neq 0$ we get that $I_H = -\frac{1}{\lambda} K \in \mathcal{K}(H)$ which is impossible since H is infinite dimensional. So we must have $\lambda = 0$. This implies that $K = 0$, hence that $(K, \lambda) = (0, 0)$, as desired. It follows that ϕ is injective.

This shows that ϕ is a $*$ -isomorphism (which is then automatically isometric).

Exercise 7. Let $\mathcal{A}$ be a C^* -algebra and $A, B \in \mathcal{A}$. Then

$$\text{sp}(AB) \cup \{0\} = \text{sp}(BA) \cup \{0\}$$

Solution: By symmetry it suffices to show that

$$\text{sp}(BA) \cup \{0\} \subseteq \text{sp}(AB) \cup \{0\}.$$

To prove this, assume that $\lambda \in \mathbb{C} \setminus (\text{sp}(AB) \cup \{0\})$. Set

$$GL(\tilde{\mathcal{A}}) := \{X \in \tilde{\mathcal{A}} \mid X \text{ is invertible in } \tilde{\mathcal{A}}\}.$$

Then $\lambda I - AB \in GL(\tilde{\mathcal{A}})$ and $\lambda \neq 0$, so $I - \frac{1}{\lambda}AB \in GL(\tilde{\mathcal{A}})$.

Set $A' := \frac{1}{\lambda}A$ and $C := (I - A'B)^{-1}$. Then $C(I - A'B) = (I - A'B)C = I$, so

$$CA'B = A'BC = C - I.$$

Further, set $D = I + BCA'$. Then we get that

$$\begin{aligned} D(I - BA') &= D - DBA' = I + BCA' - BA' - BCA'BA' \\ &= I + BCA' - BA' - B(C - I)A' \\ &= I + BCA' - BA' - BCA' + BA' \\ &= I \end{aligned}$$

Similarly we get that $(I - BA')D = I$.

This means that $I - BA' \in GL(\tilde{\mathcal{A}})$, so $\lambda I - BA = \lambda(I - BA') \in GL(\tilde{\mathcal{A}})$ since $\lambda \neq 0$. Hence $\lambda \in \mathbb{C} \setminus (\text{sp}(BA) \cup \{0\})$.

Exercise 8.

If $\mathcal{A}$ and $\mathcal{B}$ are $*$ -algebras, we denote their direct sum by $\mathcal{A} \oplus \mathcal{B}$. We recall that $\mathcal{A} \oplus \mathcal{B}$ becomes a C^* -algebra w. r. t. the norm $\|(A, B)\| := \max(\|A\|, \|B\|)$ when $\mathcal{A}$ and $\mathcal{B}$ are C^* -algebras.

a) Let $\mathcal{C}$ be a $*$ -algebra, and let $\mathcal{A}, \mathcal{B}$ be $*$ -subalgebras of $\mathcal{C}$ such that

$$\mathcal{C} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \cap \mathcal{B} = \{0\} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\}.$$

(One often says that $\mathcal{C}$ is the internal direct sum of $\mathcal{A}$ and $\mathcal{B}$ when these conditions are satisfied.)

Then $\mathcal{A}$ and $\mathcal{B}$ are $*$ -ideals in $\mathcal{C}$ and $\mathcal{C}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathcal{B}$:

Solution: Consider $A \in \mathcal{A}$, and $C \in \mathcal{C}$. Write $C = A' + B'$, with $A' \in \mathcal{A}$, $B' \in \mathcal{B}$. Then

$$CA = (A' + B')A = A'A + B'A = A'A \in \mathcal{A}.$$

We used above that $B'A \in \mathcal{B}\mathcal{A} = (\mathcal{A}\mathcal{B})^* = \{0\}$. Similarly we get that $AC \in \mathcal{A}$. Thus $\mathcal{A}$ is an ideal in $\mathcal{C}$. One shows in the same way that $\mathcal{B}$ is an ideal in $\mathcal{C}$.

To show that $\mathcal{C}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathcal{B}$, we define $\pi: \mathcal{A} \oplus \mathcal{B} \rightarrow \mathcal{C}$ by

$$\pi((A, B)) = A + B.$$

Then π is obviously surjective, and it is also injective: if $\pi((A, B)) = \pi((A', B'))$, then $A + B = A' + B'$, so $A - A' = B' - B$, where $A - A' \in \mathcal{A}$ and $B' - B \in \mathcal{B}$, so $A = A'$ and $B = B'$, since $\mathcal{A} \cap \mathcal{B} = \{0\}$.

Moreover, π is multiplicative since we have

$$\begin{aligned}\pi((A, B)(A', B')) &= AA' + BB', \text{ and} \\ \pi((A, B))\pi((A', B')) &= (A + B)(A' + B') \\ &= AA' + BA' + AB' + BB' \\ &= AA' + BB' .\end{aligned}$$

Similarly, one checks that π is linear and preserves the $*$ -operation.

b) Let $\mathcal{C}$ be a C^* -algebra and $\mathcal{A}, \mathcal{B}$ be C^* -subalgebras of $\mathcal{C}$ satisfying that

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\} .$$

Then $\mathcal{A}$ and $\mathcal{B}$ are closed $*$ -ideals in $\mathcal{C}$ and $\mathcal{C}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathcal{B}$:

Solution: We first note that we have $\mathcal{A} \cap \mathcal{B} = \{0\}$. Indeed, let $D \in \mathcal{A} \cap \mathcal{B}$ (which is a C^* -subalgebra of $\mathcal{C}$). Then

$$D = R + iS$$

where $R := \operatorname{Re}(D) = (D + D^*)/2$ and $S := \operatorname{Im}(D) = (D - D^*)/2i$ are both self-adjoint elements of $\mathcal{A} \cap \mathcal{B}$. Then we have $R^2 = R \cdot R \in \mathcal{A}\mathcal{B} = \{0\}$, which implies that $R = 0$. Similarly, $S = 0$. This gives that $D = 0$.

We can now conclude that $\mathcal{C}$ is the internal direct sum of $\mathcal{A}$ and $\mathcal{B}$ (as defined in a)), so a) gives that $\mathcal{A}$ and $\mathcal{B}$ are ideals of $\mathcal{C}$ (which are closed since they are C^* -subalgebras of $\mathcal{C}$) and $\mathcal{C} \simeq \mathcal{A} \oplus \mathcal{B}$.

c) Let $\mathcal{A}$ be a unital C^* -algebra with unit e . Define $\tilde{\mathcal{A}}$ as a $*$ -algebra in the same way as we did when $\mathcal{A}$ is nonunital, and let $I = (0, 1)$ denote the unit of $\tilde{\mathcal{A}}$.

Then $\tilde{\mathcal{A}}$ is the internal direct sum of $\mathcal{A}'$ and $\mathcal{B}$, where

$$\mathcal{A}' := \mathcal{A} \times \{0\} \quad \text{and} \quad \mathcal{B} := \mathbb{C}(I - e) = \{\lambda(-e, 1) : \lambda \in \mathbb{C}\},$$

and $\tilde{\mathcal{A}}$, as a $*$ -algebra, is therefore $*$ -isomorphic to $\mathcal{A} \oplus \mathbb{C}$.

Solution: We first note that $\tilde{\mathcal{A}} = \mathcal{A}' + \mathcal{B}$. Indeed, let $(A, \lambda) \in \tilde{\mathcal{A}}$. Then

$$(A, \lambda) = (A + \lambda e, 0) + \lambda(-e, 1) \in \mathcal{A}' + \mathcal{B}.$$

Next we show that $\mathcal{A}' \cap \mathcal{B} = \{(0, 0)\}$. Assume $(A, 0) = \lambda(-e, 1)$ for some $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Then $A = -\lambda e$ and $\lambda = 0$, so $A = 0$.

Moreover, we also have that $\mathcal{A}'\mathcal{B} = \{(0, 0)\}$, because

$$(A, 0)(\lambda(-e, 1)) = \lambda(-Ae + A, 0) = \lambda(0, 0) = (0, 0)$$

for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Thus, $\tilde{\mathcal{A}}$ is the internal direct sum of $\mathcal{A}'$ and $\mathcal{B}$.

Using a), we get that $\tilde{\mathcal{A}}$ is $*$ -isomorphic to $\mathcal{A}' \oplus \mathcal{B}$. Now $\mathcal{A}'$ is $*$ -isomorphic to $\mathcal{A}$ (via the map $(A, 0) \mapsto A$) and $\mathcal{B}$ is $*$ -isomorphic to $\mathbb{C}$ (via the map $(\lambda(I - e)) \mapsto \lambda$). It follows readily that $\tilde{\mathcal{A}}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathbb{C}$ (as $*$ -algebras).