

MAT4460 - Fall 2026 - Exercises - Week 36 (31.08-04.09)

Exercise 5

Let X denote a locally compact Hausdorff space which is not compact.

Show that the unitization $\widetilde{C_0(X)}$ of $C_0(X)$ is (isometrically) $*$ -isomorphic to $C(\widetilde{X})$, where $\widetilde{X}$ denotes the one-point compactification of X .

Exercise 6

Let H be an infinite dimensional Hilbert space.

Check first that $\mathcal{K}(H) + \mathbb{C} \cdot I_H$ is a C^* -subalgebra of $\mathcal{B}(H)$. Then show that the unitization $\widetilde{\mathcal{K}(H)}$ of $\mathcal{K}(H)$ is (isometrically) $*$ -isomorphic to $\mathcal{K}(H) + \mathbb{C} \cdot I_H$.

Exercise 7. Let $\mathcal{A}$ be a C^* -algebra and $A, B \in \mathcal{A}$. Show that

$$\text{sp}(AB) \cup \{0\} = \text{sp}(BA) \cup \{0\}.$$

Exercise 8.

When $\mathcal{A}$ and $\mathcal{B}$ are $*$ -algebras, one may equip $\mathcal{A} \times \mathcal{B}$ with the pointwise operations

$$\begin{aligned} (A_1, B_1) + (A_2, B_2) &:= (A_1 + A_2, B_1 + B_2), & \lambda(A_1, B_1) &:= (\lambda A_1, \lambda B_1), \\ (A_1, B_1)(A_2, B_2) &:= (A_1 A_2, B_1 B_2) & \text{and } (A_1, B_1)^* &:= (A_1^*, B_1^*), \end{aligned}$$

and thereby obtain a $*$ -algebra, called their *direct sum* and denoted by $\mathcal{A} \oplus \mathcal{B}$, which contains (canonical copies of) $\mathcal{A}$ and $\mathcal{B}$ as ideals¹. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are C^* -algebras, then $\mathcal{A} \oplus \mathcal{B}$ becomes a C^* -algebra with respect to the norm $\|(A, B)\| := \max(\|A\|, \|B\|)$. You should have no trouble in checking these assertions.

a) Let $\mathcal{C}$ be a $*$ -algebra and $\mathcal{A}, \mathcal{B}$ be $*$ -subalgebras of $\mathcal{C}$. Define $\mathcal{A} + \mathcal{B}$ and $\mathcal{A}\mathcal{B}$ as subsets of $\mathcal{C}$ in the obvious way. Assume that

$$\mathcal{C} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \cap \mathcal{B} = \{0\} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\}.$$

(One often says that $\mathcal{C}$ is the *internal direct sum* of $\mathcal{A}$ and $\mathcal{B}$ when these conditions are satisfied.) Show that $\mathcal{A}$ and $\mathcal{B}$ are $*$ -ideals in $\mathcal{C}$ and that $\mathcal{C}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathcal{B}$.

b) Let $\mathcal{C}$ be a C^* -algebra and $\mathcal{A}, \mathcal{B}$ be C^* -subalgebras of $\mathcal{C}$. Assume that

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\}.$$

Show that $\mathcal{A}$ and $\mathcal{B}$ are closed $*$ -ideals in $\mathcal{C}$ and that $\mathcal{C}$ is $*$ -isomorphic to $\mathcal{A} \oplus \mathcal{B}$.

c) Let $\mathcal{A}$ be a unital C^* -algebra with unit e . Define $\widetilde{\mathcal{A}}$ as a $*$ -algebra in the same way as we did when $\mathcal{A}$ is nonunital. Let $I = (0, 1)$ denote the unit of $\widetilde{\mathcal{A}}$.

Show that $\widetilde{\mathcal{A}}$ is the internal direct sum of $\mathcal{A}'$ and $\mathcal{B}$, where

$$\mathcal{A}' := \mathcal{A} \times \{0\} \quad \text{and} \quad \mathcal{B} := \mathbb{C}(I - e) = \{\lambda(-e, 1) : \lambda \in \mathbb{C}\}.$$

Deduce that $\widetilde{\mathcal{A}}$, as a $*$ -algebra, is $*$ -isomorphic to $\mathcal{A} \oplus \mathbb{C}$.

(Note: Since $\mathcal{A} \oplus \mathbb{C}$ can be made into a C^* -algebra, we may then transport the C^* -norm on $\mathcal{A} \oplus \mathbb{C}$ to a C^* -norm on $\widetilde{\mathcal{A}}$, so that $\widetilde{\mathcal{A}}$ becomes a C^* -algebra also in the case where $\mathcal{A}$ is unital. However, we will usually prefer to set $\widetilde{\mathcal{A}} := \mathcal{A}$ in this case.)

¹Recall that by an ideal in an algebra we mean a two-sided ideal