

MAT4460 - Fall 2026 - Solutions - Week 37

$\mathcal{A}$ denotes a C^* -algebra and I the unit of $\tilde{\mathcal{A}}$ ($= \mathcal{A}$ if $\mathcal{A}$ is unital).

Exercise 9

Let $A, B \in \mathcal{A}_{sa}$.

a) Assume that $f, g \in C(\text{sp}(\mathcal{A}))$ are real-valued. Then we have

$$f(A) \leq g(A) \iff f \leq g \text{ on } \text{sp}(A).$$

Solution This follows readily from the observation that

$$\text{sp}(g(A) - f(A)) = \text{sp}((g - f)(A)) = (g - f)(\text{sp}(A)).$$

b) We have $-\|A\| I \leq A \leq \|A\| I$.

Solution Let $\text{id} : \text{sp}(A) \rightarrow \mathbb{R}$ be defined by $\text{id}(\lambda) = \lambda$. Since $\lambda \leq \|A\|$ for all $\lambda \in \text{sp}(A)$, we have that $\text{id} \leq \|A\| 1_{\text{sp}(A)}$ on $\text{sp}(A)$. Hence, the continuous functional calculus and a) gives that

$$A = \text{id}(A) \leq \|A\| 1_{\text{sp}(A)}(A) = \|A\| I.$$

The proof that $-\|A\| I \leq A$ is similar.

c) Let $X \in \mathcal{A}$. Then $X^*X \leq I \Leftrightarrow \|X\| \leq 1$.

Solution Set $A := X^*X \in \mathcal{A}^+$, so A is self-adjoint and $\text{sp}(A) \subseteq [0, \infty)$. Note also that we then have $\|A\| = r(A) = \sup(\text{sp}(A))$. As usual, let $\text{id} : \text{sp}(A) \rightarrow \mathbb{R}$ be defined by $\text{id}(\lambda) = \lambda$. Using a) we get

$$\begin{aligned} X^*X \leq I &\Leftrightarrow A \leq I \Leftrightarrow \text{id} \leq 1_{\text{sp}(A)} \text{ on } \text{sp}(A) \\ &\Leftrightarrow \lambda \leq 1 \text{ for all } \lambda \in \text{sp}(A) \\ &\Leftrightarrow \sup(\text{sp}(A)) \leq 1 \Leftrightarrow \|A\| \leq 1 \\ &\Leftrightarrow \|X^*X\| \leq 1 \Leftrightarrow \|X\|^2 \leq 1 \\ &\Leftrightarrow \|X\| \leq 1 \end{aligned}$$

d) Assume $0 \leq A \leq B$. Then $\|A\| \leq \|B\|$.

Solution. From b) we know that $B \leq \|B\| I$. Hence $A \leq \|B\| I$, so (with id defined as above), we get $\text{id} \leq \|B\| 1_{\text{sp}(A)}$ on $\text{sp}(A)$. Since $\|A\| \in \text{sp}(A)$ (as $A \geq 0$), plugging in $\lambda = \|A\|$ in this inequality gives that $\|A\| \leq \|B\|$.

d) Let $A, B \in \mathcal{A}^+$. Then we have $\text{sp}(AB) \subseteq [0, \infty)$. Moreover, $AB \in \mathcal{A}^+ \Leftrightarrow AB = BA$.

Solution. We first show that $\text{sp}(AB) \subseteq [0, \infty)$:

We first observe that (using a previous exercise)

$$\text{sp}(AB) \cup \{0\} = \text{sp}(A^{1/2}A^{1/2}B) \cup \{0\} = \text{sp}(A^{1/2}BA^{1/2}) \cup \{0\}$$

Next, since $A^{1/2}$ is selfadjoint and $B \geq 0$, we get that $A^{1/2}BA^{1/2} \geq 0$. These two observations give that $\text{sp}(AB) \cup \{0\} \subseteq [0, \infty)$, and the claim above clearly follows.

Next, we show that $AB \in \mathcal{A}^+ \Leftrightarrow AB = BA$:

Since we know that $\text{sp}(AB) \subseteq [0, \infty)$, it is clear that $AB \in \mathcal{A}^+$ if and only if AB is selfadjoint. As $(AB)^* = B^*A^* = BA$, this is equivalent to $AB = BA$, as asserted above.

Exercise 10

a) Let $P \in \mathcal{A}$ be a projection. Then $\text{sp}(P) \subseteq \{0, 1\}$:

Solution. As usual, let $\text{id} : \text{sp}(A) \rightarrow \mathbb{R}$ be given by $\text{id}(\lambda) = \lambda$. Since $P^2 = P = P^*$, the continuous functional calculus gives that $(\text{id}(\lambda))^2 = \text{id}(\lambda) = \overline{\text{id}(\lambda)}$, i.e., $\lambda^2 = \lambda = \bar{\lambda}$, for every $\lambda \in \text{sp}(A)$. This implies that $\lambda \in \{0, 1\}$ for every $\lambda \in \text{sp}(A)$, showing the assertion.

b) Let $A \in \mathcal{A}_{\text{sa}}$ be such that $\text{sp}(A) \subseteq [0, \varepsilon] \cup [1 - \varepsilon, 1]$ for some $\varepsilon \in (0, 1/4)$. Then there exists a projection $P \in \mathcal{A}$ such that $\|A - P\| \leq \varepsilon$:

Solution. Let $g : [0, 1] \rightarrow [0, 1]$ be any continuous function which is equal to 0 on $[0, \varepsilon]$ and equal to 1 on $[1 - \varepsilon, 1]$. Let f denote the restriction of g to $\text{sp}(A)$. Then f is a real-valued, continuous function on $\text{sp}(A)$ which satisfies that $f(\lambda)^2 = f(\lambda)$ on $\text{sp}(A)$. Hence $P := f(A)$ is a projection in $\mathcal{A}$, and we have $\|A - P\| = \sup\{|\lambda - f(\lambda)| : \lambda \in \text{sp}(A)\} \leq \varepsilon$, as desired.

c) Assume $\mathcal{A}$ is unital. Then $\mathcal{A}$ contains no other projections than 0 and I if and only if the spectrum of every self-adjoint element of $\mathcal{A}$ is connected.

Solution. Assume first that $\mathcal{A}$ contains a projection P such that $0 \neq P \neq I$. Then $\text{sp}(P) \subseteq \{0, 1\}$, $\text{sp}(P) \neq \{0\}$ and $\text{sp}(P) \neq \{1\}$, i.e., $\text{sp}(P) = \{0, 1\}$, which is disconnected.

On the other hand, assume that $\mathcal{A}$ contains a self-adjoint element A such that $\text{sp}(A)$ is disconnected, i.e., there exists a nonempty, proper subset X of $\text{sp}(A)$ which is open and closed. Then the indicator function $f = 1_X^{\text{sp}(A)}$ of X in $\text{sp}(A)$ is a continuous function on $\text{sp}(A)$. As $f \cdot f = f = \bar{f}$, $P := f(A)$ is a projection in $\mathcal{A}$, and we have that $\text{sp}(P) = f(\text{sp}(A)) = \{0, 1\}$, so $0 \neq P \neq I$.

Exercise 11

a) Describe an approximate unit $\{e_n\}_{n \in \mathbb{N}}$ for $C_0(\mathbb{R})$.

Solution. For each $n \in \mathbb{N}$, pick any function $e_n \in C_0(\mathbb{R})$ that takes its values in $[0, 1]$, is equal to 1 on $[-n, n]$, and is equal to 0 outside $[-(n+1), n+1]$. Then $\{e_n\}_{n \in \mathbb{N}}$ is an approximate unit for $C_0(\mathbb{R})$:

It is obvious that $\{e_n\}$ is an increasing sequence of non-negative functions in $C_0(\mathbb{R})$, all having sup-norm equal to 1. So let $f \in C_0(\mathbb{R})$. We have to show that $\lim_{n \rightarrow \infty} \|f - f e_n\|_\infty = 0$.

Let $\varepsilon > 0$. We can then pick a compact subset $K \subseteq \mathbb{R}$ such that $|f(x)| < \varepsilon$ for all $x \in \mathbb{R} \setminus K$. Consider $n \in \mathbb{N}$. For $x \in \mathbb{R}$, note that

$$(f - f e_n)(x) = \begin{cases} 0 & \text{if } |x| \leq n, \\ f(x)(1 - e_n(x)) & \text{if } n < |x| \leq n+1, \\ f(x) & \text{if } |x| > n+1. \end{cases}$$

Thus, $\|f - f e_n\|_\infty = \sup\{|f(x)| : |x| > n\}$.

Now choose $N \in \mathbb{N}$ such that $K \subseteq [-N, N]$. Then for any $n \geq N$ we have $K \subseteq [-n, n]$, so $x \notin K$ whenever $|x| > n$. Thus we get that

$$\|f - f e_n\|_\infty \leq \sup\{|f(x)| : x \in \mathbb{R} \setminus K\} \leq \varepsilon$$

for all $n \geq N$. It follows that $\|f - f e_n\|_\infty \rightarrow 0$, as desired.

b) Let H be an infinite dimensional Hilbert space, and let $\{u_j\}_{j \in J}$ be an orthonormal basis for H . Set $\mathcal{F} := \{F \subseteq J : F \text{ is finite}\}$, and let $\mathcal{F}$ be directed by inclusion. For each $F \in \mathcal{F}$, let $P_F : H \rightarrow H$ denote the orthogonal projection onto $\text{Span}\{u_j : j \in F\}$.

Then the net $\{P_F\}_{F \in \mathcal{F}}$ is an approximate unit for $\mathcal{K}(H)$:

Solution. It is obvious that this net satisfies the first two properties of an approximate unit. We have therefore only to show that, for every $K \in \mathcal{K}(H)$, we have

$$\lim_F \|K - P_F K\| = \lim_F \|K - K P_F\| = 0$$

Let $K \in \mathcal{K}(H)$ and $\varepsilon > 0$. We can then find a bounded finite-rank operator K_0 on H such that $\|K - K_0\| < \frac{\varepsilon}{3}$.

Since K_0 has finite rank, there exist¹ $\xi_1, \dots, \xi_n \in H$ and $\eta_1, \dots, \eta_n \in H \setminus \{0\}$ such that

$$K_0(x) = \sum_{k=1}^n \langle x, \eta_k \rangle \xi_k \quad \text{for all } x \in H.$$

Thus for all $x \in H$ we get

$$\begin{aligned} \|(K_0 - P_F K_0)(x)\| &= \left\| \sum_{k=1}^n \langle x, \eta_k \rangle (\xi_k - P_F(\xi_k)) \right\| \\ &\leq \sum_{k=1}^n |\langle x, \eta_k \rangle| \|\xi_k - P_F(\xi_k)\| \\ &\leq \sum_{k=1}^n \|x\| \|\eta_k\| \|\xi_k - P_F(\xi_k)\| \\ &= \left(\sum_{k=1}^n \|\eta_k\| \|\xi_k - P_F(\xi_k)\| \right) \|x\|. \end{aligned}$$

Now, for every $\xi \in H$, we have that

$$\lim_F \|\xi - P_F \xi\| = 0.$$

So we can find $F_0 \in \mathcal{F}$ such that for all $F \in \mathcal{F}$ satisfying $F_0 \subseteq F$ we have

$$\|\xi_k - P_F(\xi_k)\| < \frac{\varepsilon}{3n \|\eta_k\|}$$

for every $k = 1, \dots, n$. Then, for any F as above, we get

$$\|(K_0 - P_F K_0)(x)\| \leq \left(\sum_{k=1}^n \|\eta_k\| \frac{\varepsilon}{3n \|\eta_k\|} \right) \|x\| = \frac{\varepsilon}{3} \|x\|$$

for all $x \in H$, hence $\|K_0 - P_F K_0\| \leq \frac{\varepsilon}{3}$.

Thus, for any $F \in \mathcal{F}$ satisfying $F_0 \subseteq F$, we obtain that

$$\begin{aligned} \|K - P_F K\| &\leq \|K - K_0\| + \|K_0 - P_F K_0\| + \|P_F K_0 - P_F K\| \\ &\leq \|K - K_0\| + \|K_0 - P_F K_0\| + \|P_F\| \|K_0 - K\| \\ &\leq 2\|K_0 - K\| + \|K_0 - P_F K_0\| < 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows that $\lim_F \|K - P_F K\| = 0$.

By using that the adjoint operation is isometric it follows easily that we also have $\lim_F \|K - K P_F\| = 0$ for all $K \in \mathcal{K}(H)$.

¹If you have not seen this fact before, you should prove it; if $K_0 \neq 0$, you can start by picking an o.n.b. for the range of K_0 .

Exercise 12

Assume $\mathcal{A}$ is separable (i.e., contains a countable norm-dense subset). Then there exists a sequence in $\mathcal{A}$ which is an approximate unit for $\mathcal{A}$:

Solution. Let $\{A_n : n \in \mathbb{N}\}$ be a dense subset of $\mathcal{A}$, and let $\{E_\alpha\}_{\alpha \in \Lambda}$ be an approximate unit for $\mathcal{A}$. Pick $\alpha_1 \in \Lambda$. Then pick recursively α_k for each $k \in \mathbb{N}$, $k \geq 2$, such that $\alpha_k \geq \alpha_{k-1}$ and $\|E_{\alpha_k} A_n - A_n\| < 1/k$, $\|A_n E_{\alpha_k} - A_n\| < 1/k$ for $n = 1, \dots, k$. Then it is routine to verify (using an $\varepsilon/3$ -argument) that $\{E_{\alpha_k}\}_{k \in \mathbb{N}}$ is an approximate unit for $\mathcal{A}$.