

MAT4460 - Fall 2026 - Exercises - Week 37 (07.09-11.09)

We let $\mathcal{A}$ denote a C^* -algebra. Recall that I denote the unit of $\tilde{\mathcal{A}}$ and that $\tilde{\mathcal{A}} := \mathcal{A}$ when $\mathcal{A}$ is unital.

Exercise 9

Let $A, B \in \mathcal{A}_{\text{sa}}$. Note that some of the statements below make sense in $(\tilde{\mathcal{A}})_{\text{sa}}$.

a) Assume $f, g \in C(\text{sp}(A))$ are real-valued (so $f(A)$ and $g(A)$ are self-adjoint in $\tilde{\mathcal{A}}$). Show that

$$f(A) \leq g(A) \iff f \leq g \text{ on } \text{sp}(A).$$

b) Show that $-\|A\| I \leq A \leq \|A\| I$.

c) Let $X \in \mathcal{A}$. Show that $X^*X \leq I \Leftrightarrow \|X\| \leq 1$.

d) Assume $0 \leq A \leq B$. Show that $\|A\| \leq \|B\|$.

e) Assume $A, B \geq 0$. Show that $\text{sp}(AB) \subseteq [0, \infty)$. Then show that $AB \in \mathcal{A}^+ \Leftrightarrow AB = BA$.

Exercise 10

Recall that an element $P \in \mathcal{A}$ is called an (orthogonal) *projection* if $P^2 = P = P^*$.

a) Let $P \in \mathcal{A}$ be a projection. Show that $\text{sp}(P) \subseteq \{0, 1\}$.

b) Let $A \in \mathcal{A}_{\text{sa}}$ be such that $\text{sp}(A) \subseteq [0, \varepsilon] \cup [1 - \varepsilon, 1]$ for some $\varepsilon \in (0, 1/4)$. Show that there exists a projection $P \in \mathcal{A}$ such that $\|A - P\| \leq \varepsilon$.

c) Assume $\mathcal{A}$ is unital. Show that $\mathcal{A}$ contains no other projections than 0 and I if and only if the spectrum of every self-adjoint element of $\mathcal{A}$ is connected.

Exercise 11

a) Describe an approximate unit $\{e_n\}_{n \in \mathbb{N}}$ for $C_0(\mathbb{R})$.

b) Let H be an infinite-dimensional Hilbert space, and let $\{u_j\}_{j \in J}$ be an orthonormal basis for H . Set $\mathcal{F} := \{F \subseteq J : F \text{ is finite}\}$, and let $\mathcal{F}$ be directed by inclusion. For each $F \in \mathcal{F}$, let $P_F : H \rightarrow H$ denote the orthogonal projection onto $\text{Span}\{u_j : j \in F\}$.

Show that $\{P_F\}_{F \in \mathcal{F}}$ is an approximate unit for $\mathcal{K}(H)$.

Exercise 12

Assume $\mathcal{A}$ is separable (i.e., contains a countable dense subset). Show that there exists a sequence in $\mathcal{A}$ which gives an approximate unit for $\mathcal{A}$.