

MAT4460 - Fall 2022 - Solutions - Week 38

Exercise 13

Let $H \neq \{0\}$ be a Hilbert space.

Every $T \in \mathcal{F}(H)$ can be written as a linear combination of rank one operators:

Solution. If $T = 0$, then the assertion is obvious. Assume $T \neq 0$. Let $\{\xi_1, \dots, \xi_n\}$ be an o.n.b. for the range of T , and set $\eta_j := T^*(\xi_j)$ for each $j = 1, \dots, n$. Then, for every $x \in H$, we get that

$$T(x) = \sum_{j=1}^n \langle T(x), \xi_j \rangle \xi_j = \sum_{j=1}^n \langle x, \eta_j \rangle \xi_j.$$

By deleting the zero-terms eventually occurring in the sum above, we may assume that all η_j 's are nonzero. For $\xi, \eta \in H$, let $\xi \otimes \eta : H \rightarrow H$ denote the bounded operator given by

$$(\xi \otimes \eta)(x) = \langle x, \eta \rangle \xi \quad \text{for all } x \in H.$$

Then we have $T = \sum_{j=1}^n \xi_j \otimes \eta_j$ and each operator $\xi_j \otimes \eta_j$ has rank one. This shows the claim.

Let now $\mathcal{J}$ be a nonzero ideal in $\mathcal{B}(H)$. Then we have $\mathcal{F}(H) \subseteq \mathcal{J}$:

Solution. Using what we proved above, it is clear that it suffices to show that $\mathcal{J}$ contains every operator of the form $\xi \otimes \eta$, where $\xi, \eta \in H \setminus \{0\}$.

Pick $S \in \mathcal{J} \setminus \{0\}$. Then pick $\xi_0 \in S(H)$ with $\|\xi_0\| = 1$ and $\eta_0 \in H$ such that $S(\eta_0) = \xi_0$. Then, for every $x \in H$, we get that

$$[(\xi \otimes \xi_0)S(\eta_0 \otimes \eta)](x) = \langle x, \eta \rangle [(\xi \otimes \xi_0)S](\eta_0) = \langle x, \eta \rangle \langle \xi_0, \xi_0 \rangle \xi = (\xi \otimes \eta)(x).$$

Thus, $\xi \otimes \eta = (\xi \otimes \xi_0)S(\eta_0 \otimes \eta)$ belongs to $\mathcal{J}$, since $\mathcal{J}$ is an ideal in $\mathcal{B}(H)$.

Exercise 14

Let X be a compact Hausdorff space, E be a closed subset of X , $\phi_E : C(X) \rightarrow C(E)$ be the *-homomorphism given by $\phi_E(f) = f|_E$, so $\ker \phi_E = \{f \in C(X) : f = 0 \text{ on } E\}$.

a) $\ker \phi_E \simeq C_0(X \setminus E)$:

Solution. For each $g \in C_0(X \setminus E)$, let $\tilde{g} : X \rightarrow \mathbb{C}$ be defined by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in X \setminus E, \\ 0 & \text{if } x \in E. \end{cases}$$

Then $\tilde{g} \in \ker \phi_E$. Conversely, if $f \in \ker \phi_E$, then the restriction $\check{f}$ of f to $X \setminus E$ belongs to $C_0(X \setminus E)$. It is routine to check that the maps $g \mapsto \tilde{g}$ and $f \mapsto \check{f}$ are *-homomorphisms and inverses of each other.

b) The map $E \mapsto \ker \phi_E$ is a bijection from the set of all closed subsets of X onto the set of all closed ideals in $C(X)$:

Solution. Injectivity follows easily by using Urysohn's lemma (and the fact that every compact Hausdorff space is normal, cf. any descent book in topology). To show surjectivity, let $\mathcal{J}$ be a closed ideal in $C(X)$. Define $E := \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{J}\}$. Since $E = \bigcap_{f \in \mathcal{J}} f^{-1}(\{0\})$, E is closed in X . We claim that $\ker \phi_E = \mathcal{J}$.

• $\mathcal{J} \subseteq \ker \phi_E$: Let $f \in \mathcal{J}$. Then $f(x) = 0$ for every $x \in E$, by definition of E . Thus, $f = 0$ on E , i.e., $f \in \ker \phi_E$.

• $\ker \phi_E \subseteq \mathcal{J}$: Let $f \in \ker \phi_E$. For $\varepsilon > 0$, set $X_\varepsilon := \{x \in X : |f(x)| \geq \varepsilon\}$. Then X_ε is closed and $X_\varepsilon \cap E = \emptyset$. So for each $x \in X_\varepsilon$ there exists some $f_x \in \mathcal{J}$ such that $f_x(x) \neq 0$, and we can find an open neighborhood U_x of x such that $|f_x(y)|^2 > 0$ for every $y \in U_x$. Then $\{U_x\}_{x \in X_\varepsilon}$ is an open cover of X_ε , which is compact. Thus there exist $x_1, \dots, x_n \in X_\varepsilon$ such that $X_\varepsilon \subseteq \bigcup_{j=1}^n U_{x_j}$. Set now

$$g_\varepsilon := \sum_{j=1}^n |f_{x_j}|^2 = \sum_{j=1}^n f_{x_j} \overline{f_{x_j}} \in \mathcal{J}.$$

Then $g_\varepsilon \geq 0$ on X and $g_\varepsilon > 0$ on X_ε . Setting $g_{\min} := \inf\{g_\varepsilon(x) : x \in X_\varepsilon\}$, we have $g_{\min} > 0$ (by continuity of g_ε and compactness of X_ε). Hence, letting h_ε be the continuous function on X given by $h_\varepsilon(x) := \max(g_\varepsilon(x), g_{\min})$, we get that

$$h_\varepsilon > 0 \text{ on } X, \quad h_\varepsilon \geq g_\varepsilon \text{ on } X, \quad \text{and} \quad h_\varepsilon = g_\varepsilon \text{ on } X_\varepsilon.$$

Finally, setting $f_\varepsilon := g_\varepsilon h_\varepsilon^{-1}$, we get that

$$f_\varepsilon \in \mathcal{J} \text{ (since } g_\varepsilon \in \mathcal{J}\text{), } 0 \leq f_\varepsilon \leq 1 \text{ on } X, \text{ and } f_\varepsilon = 1 \text{ on } X_\varepsilon.$$

Thus, since

$$|f - f f_\varepsilon|(x) = \begin{cases} 0 & \text{if } x \in X_\varepsilon, \\ |f(x)| |1 - f_\varepsilon(x)| & \text{if } x \in X \setminus X_\varepsilon \end{cases} < \varepsilon \quad \text{for all } x \in X,$$

we have $\|f - f f_\varepsilon\|_\infty \leq \varepsilon$. Since $f f_\varepsilon \in \mathcal{J}$, this shows that $f \in \overline{\mathcal{J}} = \mathcal{J}$, as desired.