

MAT4460 - Fall 2026 - Exercises - Week 38 (14.09-18.09)

Exercise 13

Let $H \neq \{0\}$ be a Hilbert space, and recall that $\mathcal{F}(H)$ denotes the ideal of $\mathcal{B}(H)$ consisting of all bounded finite rank operators on H .

Show first that every $T \in \mathcal{F}(H)$ can be written as a linear combination of bounded operators having rank one. Then use this to show that if $\mathcal{J}$ is any nonzero ideal of $\mathcal{B}(H)$, then $\mathcal{F}(H) \subseteq \mathcal{J}$.

(This implies that every nonzero norm-closed ideal of $\mathcal{B}(H)$ contains $\mathcal{K}(H)$. As explained in class, this can be used to show that $\mathcal{K}(H)$ is a simple C^* -algebra.)

Exercise 14

Let X be a compact Hausdorff space and E be closed (hence compact) subset of X . Let $\phi_E : C(X) \rightarrow C(E)$ be the $*$ -homomorphism given by

$$\phi_E(f) = f|_E \text{ (the restriction of } f \text{ to } E).$$

Then $\ker \phi_E = \{f \in C(X) : f = 0 \text{ on } E\}$ is a norm-closed ideal of $C(X)$. As ϕ_E is surjective (by Tietze's extension theorem), we get that (from a result proven in class) that

$$C(X)/\ker \phi_E \simeq C(E).$$

a) Check that $\ker \phi_E \simeq C_0(X \setminus E)$.

b) Show that the map $E \mapsto \ker \phi_E$ is a bijection from the set of all closed subsets of X onto the set of all closed ideals of $C(X)$.

Extra Exercise 1

During the lecture on Tuesday Sept. 8, we had a quick look at the direct sum of two C^* -algebras (see also the notes). You should fill in the missing details.

Extra Exercise 2

You have probably seen already the following result that we briefly discussed in class. If not, you should consider it as a training exercise.

Let H be a nonzero finite-dimensional Hilbert space and set $n := \dim H \in \mathbb{N}$. We recall that $M_n(\mathbb{C})$ is a C^* -algebra with respect to its usual operations and the operator norm given by $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| \leq 1\}$.

Let $\mathcal{B} = \{u_1, \dots, u_n\}$ be an orthonormal basis for H and define $\Phi_{\mathcal{B}} : \mathcal{B}(H) \rightarrow M_n(\mathbb{C})$ by

$$\Phi_{\mathcal{B}}(T) := [T]_{\mathcal{B}} \text{ (= the matrix of } T \text{ w.r.t. } \mathcal{B})$$

for all $T \in \mathcal{B}(H)$. Check that $\Phi_{\mathcal{B}}$ is a $*$ -isomorphism, hence that $\mathcal{B}(H) \simeq M_n(\mathbb{C})$.