

MAT4460 - Fall 2026 - Solutions - Week 39

As usual, $\mathcal{A}$ denotes a C^* -algebra.

Exercise 15

Assume $\mathcal{A} \neq \{0\}$. Recall that $S(\mathcal{A}) := \{\varphi \in \mathcal{A}^* : \varphi \geq 0, \|\varphi\| = 1\}$ (= the state space of $\mathcal{A}$).

Let $A \in \mathcal{A}$. Then we have

- (a) $A^* = A$ if and only if $\varphi(A) \in \mathbb{R}$ for all $\varphi \in S(\mathcal{A})$.
- b) $A \geq 0$ if and only if $\varphi(A) \geq 0$ for all $\varphi \in S(\mathcal{A})$.

Solution. (a) We have seen (cf. p. 21 in the notes) that $(\Rightarrow)$ holds for any $\varphi \geq 0$.

Conversely, assume $\varphi(A) \in \mathbb{R}$ for all $\varphi \in S(\mathcal{A})$. Then we have

$$\varphi(A^* - A) = \overline{\varphi(A)} - \varphi(A) = \varphi(A) - \varphi(A) = 0$$

for all $\varphi \in S(\mathcal{A})$. As shown in class, cf. Corollary 10.11, part (1), this implies that $A^* - A = 0$.

(b) $(\Rightarrow)$ is obvious. To prove $(\Leftarrow)$, assume $\varphi(A) \geq 0$ for all $\varphi \in S(\mathcal{A})$. Then (a) gives that $A \in \mathcal{A}_{\text{sa}}$. Moreover, let $\lambda \in \text{sp}(A)$. As observed in the notes (right after the proof of Theorem 10.10), we can find $\psi \in S(\mathcal{A})$ such that $\psi(A) = \lambda$. Since $\psi(A) \geq 0$ (by the assumption), $\lambda \geq 0$. Thus, $\text{sp}(A) \subseteq [0, \infty)$. Hence, $A \geq 0$.

We recall that if $\mathcal{A}$ is a C^* -algebra, then a positive linear functional φ on $\mathcal{A}$ is called *faithful* when $\{A \in \mathcal{A} : \varphi(A^*A) = 0\} = \{0\}$, and a linear functional τ on $\mathcal{A}$ is said to be *tracial* when it satisfies that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$.

Exercise 16

Let X be a compact Hausdorff space and set $\mathcal{A} = C(X)$. Let μ be a finite regular Borel measure on X and let φ_μ be the associated positive linear functional on $\mathcal{A}$, i.e.,

$$\varphi_\mu(f) = \int_X f \, d\mu \quad \text{for all } f \in \mathcal{A}.$$

Set $\text{supp}(\mu) = \{x \in X \mid \mu(U) > 0 \text{ for every open neighbourhood } U \text{ of } x\}$.

Then $\text{supp}(\mu)$ is a closed subset of X , and φ_μ is faithful if and only if $\text{supp}(\mu) = X$:

Solution.

- S is closed in X :

Assume $x' \in X \setminus S$. Then there exists an open neighborhood V of x' such that $\mu(V) = 0$. Consider $y \in V$. Then V is an open neighborhood of y such that $\mu(V) = 0$, so $y \notin S$. Hence $V \subseteq X \setminus S$. Thus $X \setminus S$ is open.

- φ_μ is faithful if and only if $S = X$:

We first note that $\mu(X \setminus S) = 0$. To prove this, consider $K \subseteq X \setminus S$, K compact. For each $x' \in K$ there exists some open neighborhood $V_{x'}$ of x' such that $\mu(V_{x'}) = 0$. The family

$\{V_{x'}\}_{x' \in K}$ is then an open cover of K . By compactness of K , we can therefore find open subsets $V_1, \dots, V_n$ of X such that $K \subseteq V_1 \cup \dots \cup V_n$ and $\mu(V_j) = 0$ for $j = 1, \dots, n$. Then we get that $\mu(K) \leq \sum_{j=1}^n \mu(V_j) = 0$. So $\mu(K) = 0$. Hence, since μ is regular, we obtain that

$$\mu(X \setminus S) = \sup\{\mu(K) : K \text{ compact}, K \subseteq X \setminus S\} = 0,$$

as asserted.

Assume now that $S \neq X$. Let $y \in X \setminus S$. By Urysohn's lemma we can pick $f \in C(X)$ such that $f(y) = 1$ and $f = 0$ on S . Then $f \neq 0$ and

$$\begin{aligned} \varphi_\mu(f^*f) &= \int_X |f|^2 d\mu = \underbrace{\int_{X \setminus S} |f|^2 d\mu}_{= 0, \text{ since } \mu(X \setminus S) = 0} + \underbrace{\int_S |f|^2 d\mu}_{= 0, \text{ since } f = 0 \text{ on } S} = 0 \end{aligned}$$

so φ_μ is not faithful.

Conversely, assume $S = X$. This clearly implies that $\mu(U) > 0$ for every non-empty open subset U of X . Let $f \in C(X) \setminus \{0\}$. Then $M = \|f\|_\infty > 0$ and $U := |f|^{-1}((M/2, \infty))$ is a non-empty open subset of X . Hence $\mu(U) > 0$ and

$$\varphi(f^*f) = \int_X |f|^2 d\mu \geq \int_U |f|^2 d\mu \geq \frac{M^2}{4} \mu(U) > 0.$$

This shows that $\varphi(f^*f) \neq 0$ whenever $f \neq 0$. Thus φ_μ is faithful.

Exercise 17

Let φ be a positive linear functional on $\mathcal{A}$, and set $\mathcal{L}_\varphi := \{A \in \mathcal{A} : \varphi(A^*A) = 0\}$ ¹.

a) $\mathcal{L}_\varphi$ is a closed left ideal in $\mathcal{A}$:

Solution. We first note that the Cauchy-Schwarz inequality for φ , i.e.,

$$|\varphi(B^*A)|^2 \leq \varphi(A^*A)\varphi(B^*B) \quad \text{for all } A, B \in \mathcal{A},$$

gives that $\mathcal{L}_\varphi := \{A \in \mathcal{A} : \varphi(B^*A) = 0 \text{ for all } B \in \mathcal{A}\}$. For each $B \in \mathcal{A}$, let $\varphi_B \in \mathcal{A}^*$ be given by $\varphi_B(A) := \varphi(B^*A)$ for all $A \in \mathcal{A}$. We then have $\mathcal{L}_\varphi = \bigcap_{B \in \mathcal{A}} \ker(\varphi_B)$, and we see that $\mathcal{L}_\varphi$ is a closed subspace of $\mathcal{A}$.

Finally, let $C \in \mathcal{A}$ and $A \in \mathcal{L}_\varphi$. Then for every $B \in \mathcal{A}$ we have

$$\varphi(B^*CA) = \varphi((C^*B)^*A) = 0,$$

showing that $CA \in \mathcal{L}_\varphi$. Thus, $\mathcal{L}_\varphi$ is a left ideal in $\mathcal{A}$.

b) φ is tracial if and only if $\varphi(A^*A) = \varphi(AA^*)$ for every $A \in \mathcal{A}$:

Solution. The implication $(\Rightarrow)$ is trivial. So assume that the RHS holds. We have to show that $\tau(BC) = \tau(CB)$ for all $B, C \in \mathcal{A}$. Since τ is linear and every element in $\mathcal{A}$ is a linear combination of positive elements in $\mathcal{A}$, it suffices to show that this equality holds whenever $B, C \in \mathcal{A}^+$. Setting then $A = B + iC$, we have

$$A^*A = B^2 + C^2 + i(BC - CB) \quad \text{and} \quad AA^* = B^2 + C^2 + i(CB - BC).$$

¹In class, we denoted $\mathcal{L}_\varphi$ by $\mathcal{N}$.

Since $\varphi(A^*A) = \varphi(AA^*)$, we get that $\varphi(BC - CB) = \varphi(CB - BC)$, which implies that $\varphi(BC) = \varphi(CB)$, as desired.

c) Assume, in addition, that φ is tracial. Then $\mathcal{L}_\varphi$ is a closed ideal in $\mathcal{A}$:

Solution. In view of a) it suffices to check that $AC \in \mathcal{L}_\varphi$ whenever $A \in \mathcal{L}_\varphi$, $C \in \mathcal{A}$, which follows because we then have

$$\varphi(B^*AC) = \varphi(CB^*A) = \varphi((BC^*)^*A) = 0$$

for all $B \in \mathcal{A}$ (using the tracial property of φ for the first equality).

d) Assume that $\mathcal{A}$ is unital and let τ be a linear functional on $\mathcal{A}$. Then τ is tracial if and only if $\tau(UAU^*) = \tau(A)$ for all $A \in \mathcal{A}$ and all unitaries U in $\mathcal{A}$:

Solution. We denote the set of all unitaries in $\mathcal{A}$ by $\mathcal{U}(\mathcal{A})$.

($\Rightarrow$) If τ is tracial, then $\tau(UAU^*) = \tau(AU^*U) = \tau(A)$ for all $A \in \mathcal{A}$ and $U \in \mathcal{U}(\mathcal{A})$.

($\Leftarrow$) Assume the RHS holds. We recall² that any $B \in \mathcal{A}$ is a linear combination of unitaries in $\mathcal{A}$. Hence, to show that the LHS holds, it suffices to show that $\tau(AU) = \tau(UA)$ whenever $A \in \mathcal{A}$ and $U \in \mathcal{U}(\mathcal{A})$. Using the assumption (with AU instead of A) we get

$$\tau(UA) = \tau(U(AU)U^*) = \tau(AU)$$

for all $A \in \mathcal{A}$ and $U \in \mathcal{U}(\mathcal{A})$, as desired.

Exercise 18.

Set $\mathcal{A} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and let Tr be the trace on $\mathcal{A}$, which is given by

$$\text{Tr}(A) = \sum_{j=1}^n a_{jj}$$

for all $A = [a_{ij}] \in \mathcal{A}$. For each $S \in \mathcal{A}$, let φ_S be the linear functional on $\mathcal{A}$ given by

$$\varphi_S(A) = \text{Tr}(AS) = \text{Tr}(SA)$$

for all $A \in \mathcal{A}$. We recall that $\|\varphi_S\| = \text{Tr}(|S|)$.

a) Let $S \in \mathcal{A}$. Then $\varphi_S \geq 0$ if and only if $S \in \mathcal{A}^+$, in which case we have $\|\varphi_S\| = \text{Tr}(S)$:

Assume that $S \geq 0$. Then for all $A \in \mathcal{A}$ we have

$$\begin{aligned} \varphi_S(A^*A) &= \text{Tr}(SA^*A) = \text{Tr}(S^{1/2}S^{1/2}A^*A) \\ &= \text{Tr}(S^{1/2}A^*AS^{1/2}) = \text{Tr}((AS^{1/2})^*AS^{1/2}) \geq 0. \end{aligned}$$

Hence $\varphi_S \geq 0$.

Conversely, assume that $\varphi_S \geq 0$. Then Tr and φ_S are $*$ -preserving, so we have

$$\text{Tr}(AS^*) = \text{Tr}(SA^*)^* = \overline{\text{Tr}(SA^*)} = \overline{\varphi_S(A^*)} = \varphi_S(A) = \text{Tr}(AS) \quad \text{for all } A \in \mathcal{A}.$$

²If you have not seen this fact in a previous course, you should prove it; this is a nice application of the continuous functional calculus.

Hence,

$$\mathrm{Tr}(A(S^* - S)) = \mathrm{Tr}(AS^*) - \mathrm{Tr}(AS) = \mathrm{Tr}(AS) - \mathrm{Tr}(AS) = 0 \quad \text{for all } A \in \mathcal{A}.$$

In particular, we get $\mathrm{Tr}((S^* - S)^*(S^* - S)) = 0$. Since Tr is faithful, this implies that $S^* - S = 0$, i.e., S is self-adjoint.

Next we show that $\mathrm{sp}(S) \subseteq [0, \infty)$. Pick an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ of eigenvectors for S with associated eigenvalues $\lambda_1, \dots, \lambda_n$, so we have $S\xi_j = \lambda_j\xi_j$ for $j = 1, \dots, n$. We can then write

$$S = \sum_{j=1}^n \lambda_j P_j,$$

where each P_j denote the standard matrix of the orthonormal projection from $\mathbb{C}^n$ onto $\mathrm{span}(\xi_j)$. Since $P_j \geq 0$ for $j = 1, \dots, n$, we have $\varphi_S(P_j) \geq 0$ for $j = 1, \dots, n$. Moreover, for each j , we have

$$\mathrm{Tr}(P_j) = \sum_{i=1}^n \langle P_j \xi_i, \xi_i \rangle = \langle \xi_j, \xi_j \rangle = 1.$$

Notice also that $SP_j = (\sum_{i=1}^n \lambda_i P_i)P_j = \lambda_j P_j$. Thus we get

$$\lambda_j = \lambda_j \mathrm{Tr}(P_j) = \mathrm{Tr}(\lambda_j P_j) = \mathrm{Tr}(SP_j) = \varphi_S(P_j) \geq 0$$

for $j = 1, \dots, n$. This shows that $\mathrm{sp}(S) \subseteq [0, \infty)$. Hence $S \geq 0$, as desired.

Finally, we note that if $\varphi_S \geq 0$, i.e., $S \geq 0$, then $|S| = S$, so we get that

$$\|\varphi_S\| = \mathrm{Tr}(|S|) = \mathrm{Tr}(S).$$

b) Assume that φ is a tracial positive linear functional on $\mathcal{A}$. Then $\varphi = c \mathrm{Tr}$ for some $c \geq 0$:

By a) we can write $\varphi = \varphi_S$ for some $S = [s_{ij}] \in \mathcal{A}^+$. Since φ is tracial, we have $\varphi_S(AB) = \varphi_S(BA)$, hence

$$\mathrm{Tr}(BSA) = \mathrm{Tr}(SAB) = \mathrm{Tr}(SBA) = \mathrm{Tr}(BAS), \quad \text{that is,}$$

$$\mathrm{Tr}(B(SA - AS)) = 0 \quad \text{for all } A, B \in \mathcal{A}.$$

In particular $\mathrm{Tr}((SA - AS)^*(SA - AS)) = 0$ for all $A \in \mathcal{A}$. Hence, $SA = AS$ for all $A \in \mathcal{A}$, that is, S lies in the *center* $\mathcal{Z}$ of $\mathcal{A}$.

Now it is not difficult to show that $\mathcal{Z} = \mathbb{C}I$, where I denotes the identity matrix in $\mathcal{A}$:

Indeed, for $k, l \in \{1, \dots, n\}$, let $E_{kl} = [(E_{kl})_{ij}] \in M_n(\mathbb{C})$ be the matrix given by

$$(E_{kl})_{ij} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, \dots, n\}$.³ Assume that $S = [s_{ij}] \in \mathcal{Z}$. Then

$$SE_{kl} = \begin{bmatrix} 0 & \dots & \text{(column nr. } l) & \dots & 0 \\ 0 & \dots & s_{k1} & \dots & 0 \\ 0 & \dots & s_{k2} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & s_{kn} & \dots & 0 \end{bmatrix} \quad \text{and} \quad E_{kl}S = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{1l} & s_{2l} & \dots & s_{nl} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{(row nr. } k)$$

³The matrices E_{kl} , $1 \leq k, l \leq n$, are called the *matrix units* of $M_n(\mathbb{C})$.

As $SE_{kl} = E_{kl}S$ for all $k, l \in \{1, \dots, n\}$, we get that for $k, l \in \{1, \dots, n\}$ we have

$$s_{kl} = \begin{cases} s_{kk} & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Thus, with $c := s_{11}$, we get that

$$S = \begin{bmatrix} c & 0 & \dots & 0 \\ 0 & c & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c \end{bmatrix} = cI.$$

Thus, we now know that $S \geq 0$ and $S = cI$ for some $c \in \mathbb{C}$. This gives that $c \geq 0$ and $\varphi = \varphi_S = \varphi_{cI} = c \operatorname{Tr}$, as desired.