

MAT4460 - Fall 2026 - Exercises - Week 39 (21.09-25.09)

As usual, $\mathcal{A}$ denotes a C^* -algebra.

Exercise 15

Assume $\mathcal{A} \neq \{0\}$. Recall that $S(\mathcal{A}) := \{\varphi \in \mathcal{A}^* : \varphi \geq 0, \|\varphi\| = 1\}$ (= the state space of $\mathcal{A}$).

Let $A \in \mathcal{A}$. Show that

- $A^* = A$ if and only if $\varphi(A) \in \mathbb{R}$ for all $\varphi \in S(\mathcal{A})$.
- $A \geq 0$ if and only if $\varphi(A) \geq 0$ for all $\varphi \in S(\mathcal{A})$.

In the exercises below, we use the following terminology.

- A positive linear functional φ on $\mathcal{A}$ is called *faithful* when $\{A \in \mathcal{A} : \varphi(A^*A) = 0\} = \{0\}$.
- A linear functional τ on $\mathcal{A}$ is said to be *tracial* when it satisfies that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$.

Of course, if $\mathcal{A}$ is commutative, then every linear functional on $\mathcal{A}$ is tracial.

Exercise 16

Let X be a compact Hausdorff space and set $\mathcal{A} = C(X)$. Let μ be a finite regular Borel measure on X and let φ_μ be the associated positive linear functional on $\mathcal{A}$, i.e.,

$$\varphi_\mu(f) = \int_X f \, d\mu \quad \text{for all } f \in \mathcal{A}.$$

Set $\text{supp}(\mu) = \{x \in X \mid \mu(U) > 0 \text{ for every open neighbourhood } U \text{ of } x\}$. (This set is usually called the *support* of the measure μ .)

Show that $\text{supp}(\mu)$ is a closed subset of X , and that φ_μ is faithful if and only if $\text{supp}(\mu) = X$.

Exercise 17

Let φ be a positive linear functional on $\mathcal{A}$, and set $\mathcal{L}_\varphi := \{A \in \mathcal{A} : \varphi(A^*A) = 0\}$.

a) Show that $\mathcal{L}_\varphi$ is a norm-closed *left* ideal of $\mathcal{A}$.

(*Hint*: use the Cauchy-Schwarz inequality for φ in a suitable way.)

b) Show that φ is tracial if and only if $\varphi(A^*A) = \varphi(AA^*)$ for every $A \in \mathcal{A}$.

c) Assume, in addition, that φ is tracial. Check that $\mathcal{L}_\varphi$ is then a norm-closed ideal of $\mathcal{A}$.

d) Assume that $\mathcal{A}$ is unital and let τ be a linear functional on $\mathcal{A}$. Show that τ is tracial if and only if $\tau(UAU^*) = \tau(A)$ for all $A \in \mathcal{A}$ and all unitaries U in $\mathcal{A}$.

Exercise 18.

Set $\mathcal{A} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. The linear functional Tr on $\mathcal{A}$ defined by

$$\text{Tr}(A) = \sum_{j=1}^n a_{jj}$$

for all $A = [a_{ij}] \in \mathcal{A}$ is called the *trace* on $\mathcal{A}$. Equivalently, $\text{Tr}(A) = \sum_{j=1}^n \langle Ae_j, e_j \rangle$ where $\{e_1, \dots, e_n\}$ denotes the standard basis of $\mathbb{C}^n$. Since $M_n(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^n)$, we could equally well have considered the trace $\text{Tr} : \mathcal{B}(\mathbb{C}^n) \rightarrow \mathbb{C}$, defined by

$$\text{Tr}(T) = \sum_{j=1}^n \langle T(e_j), e_j \rangle$$

for all $T \in \mathcal{B}(\mathbb{C}^n)$. We recall that the trace is more generally defined on the space of all trace-class operators on a Hilbert space H , a space which coincides with $\mathcal{B}(\mathbb{C}^n)$ when $H = \mathbb{C}^n$. The trace Tr on $\mathcal{A}$ enjoys the following properties – you should be familiar with these from MAT4450:

- Tr is a faithful tracial positive linear functional on $\mathcal{A}$.
- If $\{\xi_1, \dots, \xi_n\}$ is any o.n.b. for $\mathbb{C}^n$ and $A \in \mathcal{A}$, then $\text{Tr}(A) = \sum_{j=1}^n \langle A\xi_j, \xi_j \rangle$.
- For each $S \in \mathcal{A}$ the linear functional φ_S on $\mathcal{A}$ given by $\varphi_S(A) = \text{Tr}(AS)$ for all $A \in \mathcal{A}$ is bounded, with $\|\varphi_S\| = \text{Tr}(|S|)$.
- The map $S \rightarrow \varphi_S$ is a vector space isomorphism from $\mathcal{A}$ onto $\mathcal{A}^*$ (the dual space of $\mathcal{A}$).

a) Let $S \in \mathcal{A}$. Check that

$$\varphi_S \geq 0 \text{ if and only if } S \in \mathcal{A}^+,$$

in which case we have $\|\varphi_S\| = \text{Tr}(S)$.

b) Assume that φ is a tracial positive linear functional on $\mathcal{A}$. Show that

$$\varphi = c \text{ Tr}$$

for some $c \geq 0$ (i.e., φ is uniquely determined, up to a nonnegative scalar).

Deduce that $\mathcal{A}$ has exactly one tracial state, namely $\text{tr} := \frac{1}{n} \text{Tr}$.