

MAT4460 - Fall 2026 - Solutions - Week 41 (05.10-09.10)

If $\mathcal{A}$ is a C^* -algebra and φ is a positive linear functional on $\mathcal{A}$, then $(H_\varphi, \pi_\varphi, \xi_\varphi)$ denotes the GNS-triple associated to $\mathcal{A}$. We also recall that $\mathcal{N} := \{C \in \mathcal{A} : \varphi(C^*C) = 0\}$.

Exercise 23

Let $\mathcal{A} \neq \{0\}$ be C^* -algebra and $\pi = \oplus_{\varphi \in S(\mathcal{A})} \pi_\varphi$ denote the universal representation of $\mathcal{A}$ on the Hilbert space $H = \oplus_{\varphi \in S(\mathcal{A})} H_\varphi$ (where $S(\mathcal{A})$ denotes the state space of $\mathcal{A}$). Set $\mathcal{B} := \pi(\mathcal{A}) \simeq \mathcal{A}$ and let ψ be a positive linear functional on $\mathcal{B}$.

Then there exists a vector $\eta \in H$ such that

$$\psi(B) = \langle B\eta, \eta \rangle \quad \text{for all } B \in \mathcal{B}.$$

Solution. We may assume that $\psi \neq 0$. Let $\phi : \mathcal{A} \rightarrow \mathbb{C}$ be given by $\phi(A) = \frac{1}{\|\psi\|} \psi(\pi(A))$ for each $A \in \mathcal{A}$. Since π is injective one easily checks that ϕ is a state on $\mathcal{A}$. Let then $\eta = \{\eta_\varphi\}_{\varphi \in S(\mathcal{A})}$ be the vector in H given for each $\varphi \in S(\mathcal{A})$ by

$$\eta_\varphi = \begin{cases} \|\psi\|^{1/2} \xi_\phi & \text{if } \varphi = \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $B \in \mathcal{A}$. Then $B = \pi(A)$ for some $A \in \mathcal{A}$. Thus, $B = \oplus_{\varphi \in S(\mathcal{A})} \pi_\varphi(A)$, and $B\eta = \{\pi_\varphi(A)\eta_\varphi\}_{\varphi \in S(\mathcal{A})}$. Hence,

$$\langle B\eta, \eta \rangle = \sum_{\varphi \in S(\mathcal{A})} \langle \pi_\varphi(A)\eta_\varphi, \eta_\varphi \rangle = \|\psi\| \langle \pi_\phi(A)\xi_\phi, \xi_\phi \rangle = \|\psi\| \phi(A) = \psi(\pi(A)) = \psi(B),$$

as desired.

Exercise 24

Let $\mathcal{A}$ be a C^* -algebra and φ be a positive linear functional on $\mathcal{A}$. Assume that α is a $*$ -automorphism of $\mathcal{A}$ (that is, a $*$ -isomorphism of $\mathcal{A}$ into itself), which satisfies that $\varphi(\alpha(A)) = \varphi(A)$ for all $A \in \mathcal{A}$.

Then the linear map $V_0 : \mathcal{A}/\mathcal{N} \rightarrow \mathcal{A}/\mathcal{N}$ defined by

$$V_0(A + \mathcal{N}) = \alpha(A) + \mathcal{N}$$

extends to a unitary operator V on H_φ satisfying that

$$\pi_\varphi(\alpha(A)) = V\pi_\varphi(A)V^*$$

for all $A \in \mathcal{A}$. (This shows that $\pi_\varphi \circ \alpha \simeq \pi_\varphi$.)

Solution. Using the α -invariance of φ , one computes easily that V_0 is isometric on $\mathcal{A}/\mathcal{N}$. So V_0 extends to an isometry V on H_φ . Similarly, using that φ is clearly also α^{-1} -invariant, we get that there exists an isometry W on H_φ , satisfying that $W(A + \mathcal{N}) = \alpha^{-1}(A) + \mathcal{N}$ on $\mathcal{A}/\mathcal{N}$. It is straightforward to see that $VW = WV = I_{H_\varphi}$. Hence V is unitary, with $V^* = W$.

We check the final assertion. Let $A, B \in \mathcal{A}$. Then

$$\begin{aligned} (V\pi_\varphi(A)V^*)(B + \mathcal{N}) &= (V\pi_\varphi(A))(\alpha^{-1}(B) + \mathcal{N}) \\ &= V(A\alpha^{-1}(B) + \mathcal{N}) = \alpha(A\alpha^{-1}(B)) + \mathcal{N} \\ &= \alpha(A)B + \mathcal{N} = \pi_\varphi(\alpha(A))(B + \mathcal{N}) \end{aligned}$$

Since $\mathcal{A}/\mathcal{N}$ is dense in H_ϕ , this implies that $V\pi_\varphi(A)V^* = \pi_\varphi(A)$.

Exercise 25

Let H be a Hilbert space, $\mathcal{A}$ be a $*$ -subalgebra of $\mathcal{B}(H)$ and P be a projection in $\mathcal{B}(H)$, i.e., $P = P_K$ for some closed subspace K of H .

a) Assume that $P \in \mathcal{A}$.¹ Then PAP is a $*$ -subalgebra of $\mathcal{B}(H)$, which is a C^* -subalgebra if $\mathcal{A}$ is a C^* -subalgebra:

Solution. It is straightforward to verify that PAP is a $*$ -subalgebra of $\mathcal{B}(H)$ (in fact of $\mathcal{A}$). Assume that $\mathcal{A}$ is a C^* -subalgebra of $\mathcal{B}(H)$. To show that PAP is a C^* -subalgebra of $\mathcal{B}(H)$ (in fact of $\mathcal{A}$), it suffices to show that it is closed (w.r.t. operator norm). Let $B \in \mathcal{B}(H)$ and let $\{A_n\}$ be a sequence in $\mathcal{A}$ such that $\|PA_nP - B\| \rightarrow 0$ as $n \rightarrow \infty$. Then $B \in \mathcal{A}$ (since $PA_nP \in \mathcal{A}$ for every n) and we have

$$PA_nP = P(PA_nP)P \rightarrow PBP \text{ as } n \rightarrow \infty.$$

Thus we get that $B = PBP \in PAP$, as desired.

b) If $T \in \mathcal{B}(H)$, let $T_K \in \mathcal{B}(K)$ be the *compression of T to K* , which is defined by

$$T_K\xi := PT\xi \quad \text{for all } \xi \in K.$$

Then the map $PTP \mapsto T_K$ from $P\mathcal{B}(H)P$ to $\mathcal{B}(K)$ is a $*$ -isomorphism:

Solution. We first make an observation. Assume that $PTP = 0$. Then for any $\xi, \eta \in K$, we have

$$\langle T\xi, \eta \rangle = \langle TP\xi, P\eta \rangle = \langle PTP\xi, \eta \rangle = 0.$$

Thus, T maps K into $K^\perp$, which implies that $T_K\xi = PT\xi = 0$ for every $\xi \in K$, i.e., $T_K = 0$. Using this observation, it follows readily that if $PTP = PSP$, then $T_K = S_K$, i.e., the above map, say Φ , is well-defined. It is then straightforward to check that Φ is a $*$ -homomorphism from $P\mathcal{B}(H)P$ into $\mathcal{B}(K)$. If $T_K = 0$, then for all $\xi \in H$, we have $PTP\xi = T_K(P\xi) = 0$, i.e., $PTP = 0$. Thus, Φ is injective. Finally, if $S \in \mathcal{B}(K)$, and we define $T \in \mathcal{B}(H)$ by $T = J_KSP$, where $J_K : K \rightarrow H$ denotes the inclusion map, then $S = T_K$, so Φ is also surjective.

c) Assume that K is $\mathcal{A}$ -invariant. We then have $PT = TP$ for all $T \in \mathcal{A}$, i.e., $P \in \mathcal{A}'$, so $PAP = \mathcal{A}P = PA$ in this case. Set $\mathcal{A}_K := \{T_K : T \in \mathcal{A}\} \subseteq \mathcal{B}(K)$.

Then $\mathcal{A}_K$ is a $*$ -subalgebra of $\mathcal{B}(K)$, and the map

$$PTP \mapsto T_K$$

from PAP to $\mathcal{A}_K$ is a $*$ -isomorphism:

¹Sorry, this (obviously necessary) assumption was missing in the text!

Solution. It is almost obvious $PAP = \mathcal{A}P = P\mathcal{A}$ is a $*$ -subalgebra of $\mathcal{B}(H)$. Letting $\Phi : PB(H)P \rightarrow \mathcal{B}(K)$ denote the map from b), we get, using that Φ is a $*$ -isomorphism, that $\mathcal{A}_K = \Phi(PAP)$ is a $*$ -subalgebra of $\mathcal{B}(K)$, and the map $PTP \mapsto T_K = \Phi(PTP)$ is a $*$ -isomorphism from PAP to $\mathcal{A}_K$.

Exercise 26

Let $\mathcal{A}$ be a C^* -algebra. Recall that if S is a subset of a Hilbert space H , then we use the notation $[S] := \overline{\text{span}(S)}$.

a) Assume that $\{H_j\}_{j \in J}$ is a family of Hilbert spaces and π_j is a representation of $\mathcal{A}$ on H_j for each $j \in J$. Set $H = \bigoplus_{j \in J} H_j$ and let $\pi = \bigoplus_{j \in J} \pi_j$.

For each $k \in J$, set

$$H'_k = \{ \{\xi_j\} \in H \mid \xi_j = 0 \text{ for all } j \in J \setminus \{k\} \}.$$

Then the following facts hold:

- (i) For each $k \in J$, H'_k is π -invariant, and the restriction π'_k of π to H'_k is unitarily equivalent to π_k .
- (ii) Each H'_k is a closed subspace of H , and $H'_k \perp H'_l$ for all $k, l \in J$ such that $k \neq l$.
- (iii) $H = \left[\bigcup_{k \in J} H'_k \right]$.

Solution.

(ii) For each $k \in J$, let $P_k : H \rightarrow H$ denote the map given by $P_k(\{\xi_j\}) = \{\xi'_j\}$, where

$$\xi'_j := \begin{cases} \xi_k & \text{for } j = k, \\ 0 & \text{for } j \in J \setminus \{k\}. \end{cases}$$

Then one easily verifies that each P_k is an orthogonal projection in $\mathcal{B}(H)$ such that H'_k is the range of P_k , hence is a closed subspace of H . It is almost obvious that $P_k P_l = 0$, hence that $H'_k \perp H'_l$, whenever $k, l \in J$ are such that $k \neq l$.

(i) Let $k \in J$. For each $A \in \mathcal{A}$ one checks readily that $\pi(A)$ commutes with P_k . It follows that $H'_k = P_k(H)$ is π -invariant. Moreover, let $U_k : H_k \rightarrow H'_k$ be defined for each $\eta \in H_k$ by $U_k(\eta) = \{\xi_j\}$, where $\xi_k = \eta$ and $\xi_j = 0$ for all $j \in J \setminus \{k\}$. It is straightforward to check that U_k is a unitary operator satisfying that $U_k \pi_k(A) = \pi'_k(A) U_k$ for all $A \in \mathcal{A}$. Hence $\pi'_k \simeq \pi_k$.

(iii) Let $\xi = \{\xi_j\} \in H$ and let $\varepsilon > 0$. Since $\|\xi\|^2 = \sum_{j \in J} \|\xi_j\|^2 < \infty$, we can choose $F \subseteq J$, F finite, such that

$$\sum_{j \in J \setminus F} \|\xi_j\|^2 < \varepsilon^2.$$

Set $\xi_F = \sum_{k \in F} P_k \xi \in \text{Span}\left(\bigcup_{k \in F} H'_k\right)$. Then we have

$$\|\xi - \xi_F\| = \left(\sum_{j \in J \setminus F} \|\xi_j\|^2 \right)^{1/2} < \varepsilon.$$

It follows that $\xi \in \left[\bigcup_{k \in J} H'_k \right]$, as desired.

b) Let π be a representation of $\mathcal{A}$ on some Hilbert space H , and assume that there exists a family $\{H_j\}_{j \in J}$ of π -invariant closed subspaces of H satisfying

$$H_k \perp H_l \text{ for all } k, l \in J \text{ such that } k \neq l, \text{ and } H = \left[\bigcup_{j \in J} H_j \right].$$

For each $j \in J$, let $\pi_j : \mathcal{A} \rightarrow \mathcal{B}(H_j)$ denote the restriction of π to H_j .

Then π is unitary equivalent to $\oplus_{j \in J} \pi_j$.

Solution. We have to show that there exists a unitary operator $U : H \rightarrow \oplus_{j \in J} H_j$ such that

$$U \pi(A) U^* = (\oplus_{j \in J} \pi_j)(A) \text{ for all } A \in \mathcal{A}.$$

For each $j \in J$, let $P_j : H \rightarrow H$ denote the orthogonal projection of H on H_j . Since the ranges of these projections are pairwise orthogonal (by assumption), we have $P_j P_k = 0$ whenever $j \neq k$, so we get that for each finite $F \subseteq J$, the operator $P_F := \sum_{j \in F} P_j$ is an orthogonal projection in $\mathcal{B}(H)$.

We note that $\sum_{j \in J} P_j = I_H$ (in the SOT), i.e., $\lim_F P_F \xi = \xi$ for all $\xi \in H$:

To prove this, let $\xi \in H$ and $\varepsilon > 0$ be given. The assumption $H = \left[\bigcup_{j \in J} H_j \right]$ gives that there exists a finite $F_0 \subseteq J$ and $\eta \in P_{F_0}(H)$ such that $\|\xi - \eta\| < \varepsilon/2$. Then, for any finite $F \subseteq J$ such that $F_0 \subseteq F$, we have $P_F \eta = \eta$, so we get that

$$\|\xi - P_F \xi\| = \|\xi - \eta + P_F \eta - P_F \xi\| \leq \|\xi - \eta\| + \|P_F(\eta - \xi)\| \leq 2\|\xi - \eta\| < \varepsilon.$$

This shows that $P_F \xi \rightarrow \xi$, as desired.

Thus, for each $\xi \in H$, using Pythagoras' identity, we get that

$$\|\{P_j \xi\}\|^2 = \sum_{j \in J} \|P_j \xi\|^2 = \lim_F \sum_{j \in F} \|P_j \xi\|^2 = \lim_F \|P_F \xi\|^2 = \|\xi\|^2.$$

It follows that the map $U : H \rightarrow \oplus_{j \in J} H_j$ defined by $U\xi = \{P_j \xi\}$ for all $\xi \in H$ is well-defined, linear and isometric. From part (iii) of a), we see that the range of U contains a dense subspace of $\oplus_{j \in J} H_j$. So this range, being closed, is equal to $\oplus_{j \in J} H_j$. We can therefore conclude that U is unitary. Moreover, for all $A \in \mathcal{A}$ and $\xi \in H$, using that each P_j commutes with $\pi(A)$ (as H_j is π -invariant), we get that

$$\begin{aligned} U \pi(A) \xi &= \{P_j \pi(A) \xi\} = \{\pi(A) P_j \xi\} = \{\pi_j(A) P_j \xi\} \\ &= [(\oplus_{j \in J} \pi_j)(A)] \{P_j \xi\} = [(\oplus_{j \in J} \pi_j)(A)] U \xi, \end{aligned}$$

so U satisfies the desired property.

Exercise 27

Let $\mathcal{A}$ be C^* -algebra and π be a representation of $\mathcal{A}$ on a Hilbert space H .

Recall that $H_\pi^{\text{ess}} := [\pi(\mathcal{A})H]$, and that, because H_π^{ess} is π -invariant, we obtain a representation $\pi^{\text{ess}} := \pi|_{H_\pi^{\text{ess}}}$ of $\mathcal{A}$ on H_π^{ess} by restriction.

Let $\{E_\alpha\}$ be an approximate unit for $\mathcal{A}$.

a) $\pi^{\text{ess}}(E_\alpha) \rightarrow I_{H_\pi^{\text{ess}}}$ (in the SOT), i.e., $\pi^{\text{ess}}(E_\alpha) \xi \rightarrow \xi$ for every $\xi \in H_\pi^{\text{ess}}$:

Solution. Consider first $A \in \mathcal{A}$ and $\eta \in H$. Since $E_\alpha A \rightarrow_\alpha A$ (in operator norm) and π is continuous, we get that

$$\pi^{\text{ess}}(E_\alpha) \pi(A) \eta = \pi(E_\alpha A) \eta \rightarrow_\alpha \pi(A) \eta.$$

By linearity, it follows that $\pi^{\text{ess}}(E_\alpha) \xi \rightarrow_\alpha \xi$ for all $\xi \in \text{Span}(\pi(\mathcal{A})H)$.

Now consider $\zeta \in H_\pi^{\text{ess}}$, and let $\varepsilon > 0$. Pick $\eta \in \text{Span}(\pi(\mathcal{A})H)$ such that $\|\zeta - \eta\| < \varepsilon/3$, and choose α_0 such that $\|\pi^{\text{ess}}(E_\alpha) \eta - \eta\| < \varepsilon/3$ for all $\alpha \gtrsim \alpha_0$.

Using that $\|\pi(E_\alpha)\| \leq \|E_\alpha\| \leq 1$ for all α , and the triangle inequality, we get

$$\begin{aligned} \|\pi^{\text{ess}}(E_\alpha) \zeta - \zeta\| &\leq \|\pi^{\text{ess}}(E_\alpha) \zeta - \pi^{\text{ess}}(E_\alpha) \eta\| + \|\pi^{\text{ess}}(E_\alpha) \eta - \eta\| + \|\eta - \zeta\| \\ &\leq 2\|\eta - \zeta\| + \|\pi^{\text{ess}}(E_\alpha) \eta - \eta\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for all $\alpha \gtrsim \alpha_0$. Thus, $\pi^{\text{ess}}(E_\alpha) \zeta \rightarrow_\alpha \zeta$. This shows the assertion.

b) π is non-degenerate if and only if $\pi(E_\alpha) \rightarrow_\alpha I_H$ (in the SOT):

Solution. If π is non-degenerate, i.e., $H_\pi^{\text{ess}} = H$, then $\pi^{\text{ess}} = \pi$, and the above assertion gives that $\pi(E_\alpha) \rightarrow I_H$ (in the SOT). Conversely, if $\pi(E_\alpha) \rightarrow I_H$ (in the SOT), then for every $\xi \in H$, we have

$$\xi = \lim_\alpha \pi(E_\alpha) \xi \in [\pi(\mathcal{A})\xi] \subseteq [\pi(\mathcal{A})H] = H_\pi^{\text{ess}},$$

hence $H = H_\pi^{\text{ess}}$.

c) Does $\pi(E_\alpha)$ converge in the SOT of $\mathcal{B}(H)$ if π is degenerate?

Solution. Yes, it does. Let P denote the orthogonal projection of H on H_π^{ess} . Then we have that $\pi(E_\alpha) \rightarrow P$ (in the SOT):

Indeed, let $\xi \in H$. Then we may write

$$\xi = \zeta + \zeta',$$

where $\zeta \in H_\pi^{\text{ess}}$ and $\zeta' \in (H_\pi^{\text{ess}})^\perp = \{\eta \in H : \pi(A)\eta = 0 \text{ for all } A \in \mathcal{A}\}$. Thus, using a), we get

$$\lim_\alpha \pi(E_\alpha) \xi = \lim_\alpha \pi(E_\alpha) \zeta + \lim_\alpha \pi(E_\alpha) \zeta' = \zeta + 0 = P\xi.$$

d) Use b) to deduce that $\xi \in [\pi(\mathcal{A})\xi]$ for every $\xi \in H$ whenever π is non-degenerate.

Solution. This follows immediately from the proof of b) given above.