MAT4460 - Fall 2026 - Exercises - Week 42 (12.10-16.10)

Exercise 28

Let H be a Hilbert space.

a) Let φ be a linear functional on $\mathcal{B}(H)$. Then the following conditions are equivalent:

i) There exist $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that

$$\varphi(T) = \sum_{j=1}^{n} \langle T\xi_j, \eta_j \rangle \quad \text{for all } T \in \mathcal{B}(H).$$

ii) φ is WOT-continuous.

iii) φ is SOT-continuous.

Solution. This is actually shown in Strung's book (cf. Proposition 3.2.8). Anyhow, here is a detailed proof.

The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are immediate. We show (iii) $\Longrightarrow$ (i).

Assume φ is SOT-continuous. Since $\varphi(0) = 0$, there exists an SOT-open neighborhood U around 0 such that $\varphi(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. We can then find $\varepsilon > 0$ and $\xi_1', \dots, \xi_n' \in H$ such that

$$\{S \in B(H) : \|S\xi_j'\| < \varepsilon \text{ for } j = 1, \dots, n\} \subseteq U.$$

This means that if $S \in \mathcal{B}(H)$ and $\max_{j\in\{1,\dots,n\}} \|S\xi_j'\| < \varepsilon$, then $|\varphi(S)| < 1$.

Set $\xi_j := \frac{2}{\varepsilon}\,\xi_j'$ for $j = 1, \dots, n$ and $p(S) := \max_{j\in\{1,\dots,n\}} \|S\xi_j\|$ for $S \in \mathcal{B}(H)$.
Then we get that

$$p(S) \leq 1 \implies |\varphi(S)| < 1$$

(because $\|S\xi_j\| \leq 1 \implies \|S\xi_j'\| \leq \frac{\varepsilon}{2} < \varepsilon$).

Observe that

$$|\varphi(T)| \leq p(T) \leq \Big(\sum_{j=1}^{n} \|T\xi_j\|^2\Big)^{1/2} \quad \text{for all } T \in \mathcal{B}(H). \tag{1}$$

Indeed, if $p(T) = 0$, then $p(\lambda T) = \lambda p(T) = 0$ for all $\lambda \in \mathbb{R}^+$, so $|\varphi(\lambda T)| < 1$ for all $\lambda \in \mathbb{R}^+$, which gives that $|\varphi(T)| = 0$. So the first inequality holds in this case. Next, assume that $p(T) > 0$. Then $p(\frac{1}{p(T)}T) = 1$, so $|\varphi(\frac{1}{p(T)}T)| < 1$, that is, $|\varphi(T)| < p(T)$. Thus the first inequality holds in general. The second inequality is obvious.

Now consider $H^n = H \oplus \cdots \oplus H$ (n-times). Set

$$M_0 := \{(T\xi_1, \dots, T\xi_n) : T \in \mathcal{B}(H)\} \subseteq H^n.$$

Then M_0 is a subspace of H^n, so $M := \overline{M_0}^{\|\cdot\|}$ is a closed subspace of H^n.

Next, define $\psi\colon M_0 \to \mathbb{C}$ by

$$\psi((T\xi_1, \dots, T\xi_n)) = \varphi(T)$$

for all $T \in \mathcal{B}(H)$. Observe that ψ is well-defined: if $(T'\xi_1, \dots, T'\xi_n) = (T\xi_1, \dots, T\xi_n)$, then

$$p(T - T') = \max_{j \in \{1,\dots,n\}} \|(T - T')\xi_j\| = 0,$$

so $\varphi(T - T') = 0$ (using (1)). Clearly ψ is linear (as φ is linear). Moreover ψ is bounded: Indeed, for any $T \in \mathcal{B}(H)$, using (1) again, we get

$$|\psi((T\xi_1, \dots, T\xi_n))| = |\varphi(T)| \leq \big(\sum_{j=1}^{n} \|T\xi_j\|^2\big)^{1/2} = \|(T\xi_1, \dots, T\xi_n)\| \text{ (norm in } H^n).$$

Using the Hahn-Banach theorem we can extend ψ to $\widetilde{\psi} \in M^*$. Then the Riesz representation theorem for M^* gives that there exists $\eta = (\eta_1, \dots, \eta_n) \in M$ such that $\widetilde{\psi}(\xi) = \langle \xi, \eta \rangle$ for all $\xi \in M$. Thus, for every $T \in \mathcal{B}(H)$, we get that

$$\varphi(T) = \widetilde{\psi}((T\xi_1, \dots, T\xi_n)) = \langle (T\xi_1, \dots, T\xi_n), \overbrace{(\eta_1, \dots, \eta_n)}^{\eta} \rangle = \sum_{j=1}^{n} \langle T\xi_j, \eta_j \rangle$$

as desired.

In b) we will need to use the following result. We include a proof for completeness.

Let C be a (non-empty) convex subset of a locally convex topological vector space (X, τ). Then $x \in \overline{C}^{\tau}$ if and only if there exists a net $\{x_\alpha\}$ in C such that

$$\varphi(x) = \lim_{\alpha} \varphi(x_\alpha) \quad \textit{for all } \varphi \in (X, \tau)^*.$$

Proof. The implication ($\Rightarrow$) is straightforward. To show the converse implication, suppose that $x \in X \setminus \overline{C}^{\tau}$. We may then choose an open convex $U \subseteq X$ such that $x \in U$ and $U \cap \overline{C}^{\tau} = \emptyset$. Since $\overline{C}^{\tau}$ is closed and convex, we can apply the Hahn-Banach separation theorem and find $\varphi \in (X, \tau)^*$ and $t \in \mathbb{R}$ such that

$$\mathrm{Re}(\varphi(u)) < t \leq \mathrm{Re}(\varphi(b))$$

for all $u \in U$ and $b \in \overline{C}^{\tau}$. Thus, for any net $\{x_\alpha\}$ in C, we get that

$$\mathrm{Re}(\varphi(x)) < t \leq \mathrm{Re}(\varphi(x_\alpha)) \quad \text{for all } \alpha,$$

and this implies that $\varphi(x_\alpha) \not\to \varphi(x)$. This shows that the right hand side of the equivalence does not hold.

b) Let $\mathcal{C}$ be a (non-empty) convex subset of $\mathcal{B}(H)$. Then $\overline{\mathcal{C}}^{\mathrm{WOT}} = \overline{\mathcal{C}}^{\mathrm{SOT}}$:

Solution. Part a) says that $(\mathcal{B}(H), \mathrm{SOT})^* = (\mathcal{B}(H), \mathrm{WOT})^*$. Hence the result above gives that

$$T \in \overline{\mathcal{C}}^{\mathrm{WOT}} \Leftrightarrow T \in \overline{\mathcal{C}}^{\mathrm{SOT}},$$

which shows that the desired equality holds.

Exercise 29

Let $(X, \mathcal{B}, \mu)$ be a finite measure space. We set $L^\infty := L^\infty(X, \mathcal{B}, \mu)$ and $L^2 := L^2(X, \mathcal{B}, \mu)$. As usual, we identify $\mathcal{B}$-measurable functions which agree μ-a.e. Let M be the faithful representation of L^∞ on L^2 given by $[M(f)](\xi) = f\xi$. Set $\mathcal{M} := M(L^\infty) \subseteq \mathcal{B}(L^2)$.

a) *We have* $\mathcal{M}' = \mathcal{M}$:

Solution. Since L^∞ is abelian, it is clear that $\mathcal{M}$ is also abelian. Hence $\mathcal{M} \subseteq \mathcal{M}'$. Conversely, let $T \in \mathcal{M}'$, $T \neq 0$ (otherwise it belongs to $\mathcal{M}$). Then we have that

$$TM(f) = M(f)T$$

for all $f \in L^\infty$.
Set $g = T(1_X) \in L^2$. We claim that $g \in L^\infty$.

To show this, set $A := \{x \in X : |g(x)| > \|T\|\}$. Then $\mu(A) = 0$. Indeed, assume (for contradiction) that $\mu(A) > 0$. Observe that if $f \in L^\infty$, then

$$fg = [M(f)](g) = [M(f)T](1_X) = [TM(f)](1_X) = T(f)$$

which gives that

$$\|fg\|_2 \leq \|T\| \, \|f\|_2 \qquad (*)$$

hence that

$$\begin{aligned} \|T\|^2 \mu(A) = \int_A \|T\|^2 \, \mathrm{d}\mu \, < \int_A |g|^2 \, \mathrm{d}\mu = \int_X |\chi_A g|^2 \, \mathrm{d}\mu \\ = \|\chi_A g\|_2^2 \underset{\text{using } (*)}{\leq} \|T\|^2 \, \|\chi_A\|_2^2 = \|T\|^2 \mu(A), \end{aligned}$$

which is not possible.

This means that $|g| \leq \|T\|$ μ-a.e., so

$$\|g\|_\infty = \inf\{M > 0 : |g| \leq M \, \mu\text{-a.e.}\} \leq \|T\| < \infty,$$

which shows the claim.

We then get that

$$T(f) = fg = gf \underset{g \in L^\infty}{=} [M(g)](f) \quad \text{for all } f \in L^\infty.$$

But L^∞ is dense in L^2, because all simple $\mathcal{B}$-measurable functions belong to L^∞ and this set is dense in L^2 when μ is finite. We can therefore conclude that $T = M(g) \in \mathcal{M}$. Thus we have shown that $\mathcal{M}' \subseteq \mathcal{M}$.

Altogether, we have shown that $\mathcal{M}' = \mathcal{M}$.

b) Let $\mathcal{N}$ be an abelian von Neumann algebra on L^2 such that $\mathcal{M} \subseteq \mathcal{N}$. *Then we have* $\mathcal{M} = \mathcal{N}$ *(which shows that* $\mathcal{M}$ *is a* maximal *abelian von Neumann algebra on* L^2*)*:

Solution. Using a), we get $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{M}' = \mathcal{M} \subseteq \mathcal{N}$, hence $\mathcal{N} = \mathcal{M}$.

Exercise 30

Let H be a Hilbert space, $n \in \mathbb{N}$ and $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(H)$. We identify $M_n(\mathcal{B}(H))$ with $\mathcal{B}(H^n)$ in the canonical way. Let then $\mathcal{A}_n$ be the $*$-subalgebra of $\mathcal{B}(H^n)$ given by

$$\mathcal{A}_n = \Big\{ \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{bmatrix} : A \in \mathcal{A} \Big\}.$$

Similarly, set

$$(\mathcal{A}'')_n = \Big\{ \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{bmatrix} : B \in \mathcal{A}'' \Big\}.$$

In the course of the proof of von Neumann's double commutant theorem, we have seen that

$$(\mathcal{A}_n)' = M_n(\mathcal{A}') \subseteq \mathcal{B}(H^n) \text{ and } (\mathcal{A}'')_n \subseteq (\mathcal{A}_n)''.$$

We also have $(\mathcal{A}_n)'' \subseteq (\mathcal{A}'')_n$ *(so we can conlude that* $(\mathcal{A}'')_n = (\mathcal{A}_n)''$*):*

Solution. For $k, l \in \{1, \dots, n\}$, let $E_{kl} = \big[(E_{kl})_{ij}\big] \in M_n(\mathcal{B}(H))$ be the matrix given by

$$(E_{kl})_{ij} = \begin{cases} I_H & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, \dots, n\}$. Then, for any $T = [T_{ij}] \in M_n(\mathcal{B}(H))$, we have

$$TE_{kl} = \begin{bmatrix} & & \overset{\text{(column nr. } l\text{)}}{} & & \\ 0 & \dots & T_{k1} & \dots & 0 \\ 0 & \dots & T_{k2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & T_{kn} & \dots & 0 \end{bmatrix}, \quad E_{kl}T = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ T_{1l} & T_{2l} & \dots & T_{nl} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \text{ (row nr. } k\text{)}.$$

Assume now $T \in (\mathcal{A}_n)''$, i.e., $T \in ((\mathcal{A}_n)')' = M_n(\mathcal{A}')'$.

Let $k, l \in \{1, \dots, n\}$. Since $E_{kl} \in M_n(\mathcal{A}')$, we get that $TE_{kl} = E_{kl}T$. From the previous computation, we see that $T_{kk} = T_{ll}$ when $k = l$, while $T_{kl} = 0$ when $k \neq l$. Thus, with $S := T_{11}$, we get that

$$T = \begin{bmatrix} S & 0 & \dots & 0 \\ 0 & S & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S \end{bmatrix}.$$

Let $B \in \mathcal{A}'$ and set

$$R = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{bmatrix} \in M_n(\mathcal{A}').$$

Then we have $RT = TR$, and this clearly implies that $SB = BS$.

Since this holds for every $B \in \mathcal{A}'$, this shows that $S \in \mathcal{A}''$, hence that $T \in (\mathcal{A}'')_n$, as desired.

Exercise 31

Let H be a Hilbert space and $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(H)$. Identifying $M_2(\mathcal{B}(H))$ with $\mathcal{B}(H \oplus H)$ in the canonical way, we regard $M_2(\mathcal{A})$ as a $*$-subalgebra of $\mathcal{B}(H \oplus H)$.

Then $M_2(\overline{\mathcal{A}}^{\mathrm{SOT}}) = \overline{M_2(\mathcal{A})}^{\mathrm{SOT}}$:

Solution. It should be straightforward to check that $M_2(\overline{\mathcal{A}}^{\mathrm{SOT}})$ is SOT-closed. Since $M_2(\mathcal{A}) \subseteq M_2(\overline{\mathcal{A}}^{\mathrm{SOT}})$, this implies that $\overline{M_2(\mathcal{A})}^{\mathrm{SOT}} \subseteq M_2(\overline{\mathcal{A}}^{\mathrm{SOT}})$.

To show the reverse inclusion, consider $[B_{ij}] \in M_2(\overline{\mathcal{A}}^{\mathrm{SOT}})$. To show that $[B_{ij}] \in \overline{M_2(\mathcal{A})}^{\mathrm{SOT}}$ we have to show that, given $\varepsilon > 0$ and $\xi^k = (\xi_1^k, \xi_2^k) \in H \oplus H$ for $k = 1, \ldots, n$, there exists $[A_{ij}] \in M_2(\mathcal{A})$ such that

$$\|([B_{ij}] - [A_{ij}])\xi^k\| \ < \varepsilon \text{ for } k = 1, \ldots, n. \tag{2}$$

Now, for each k, one readily sees that

$$\|([B_{ij}] - [A_{ij}])\xi^k\| \ \leq \|(B_{11} - A_{11})\xi_1^k\| + \|(B_{12} - A_{12})\xi_2^k\| + \|(B_{21} - A_{21})\xi_1^k\| + \|(B_{22} - A_{22})\xi_2^k\|.$$

Since $B_{ij} \in \overline{\mathcal{A}}^{\mathrm{SOT}}$ for each i, j, we can find $A_{11}, A_{12}, A_{21}, A_{22} \in \mathcal{A}$ such that we have

$$\|(B_{11} - A_{11})\xi_1^k\| < \varepsilon/4, \|(B_{12} - A_{12})\xi_2^k\| < \varepsilon/4, \|(B_{21} - A_{21})\xi_1^k\| < \varepsilon/4, \|(B_{22} - A_{22})\xi_2^k\| < \varepsilon/4,$$

for each $k = 1, \ldots, n$, and it clearly follows that (2) is satisfied.

It follows that if $\mathcal{A}$ *is a von Neumann algebra on* H*, then* $M_2(\mathcal{A})$ *is a von Neumann algebra on* $H \oplus H$:

Solution. Indeed, if $\mathcal{A}$ is a von Neumann algebra, then $\overline{\mathcal{A}}^{\mathrm{SOT}} = \mathcal{A}$, so

$$\overline{M_2(\mathcal{A})}^{\mathrm{SOT}} = M_2(\overline{\mathcal{A}}^{\mathrm{SOT}}) = M_2(\mathcal{A}).$$

Moreover, since $I_H \in \mathcal{A}$, we also have $I_{H \oplus H} = \mathrm{diag}(I_H, I_H) \in M_2(\mathcal{A})$.

Exercise 32

Let H be a Hilbert space and $\{A_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}(H)$ that is WOT-convergent. Then $\{A_n\}$ is norm-bounded:

Solution. Let $\eta \in H$ and consider the sequence $(\varphi_n) \in H^*$ defined for each $n \in \mathbb{N}$ by

$$\varphi_n(\xi) = \langle \xi, A_n \eta \rangle \quad \text{ for each } \xi \in H.$$

Then, for each $\xi \in H$, the sequence $\varphi_n(\xi)$ is convergent in $\mathbb{C}$ (using the assumption), so it is bounded, i.e., we have $\sup_{n \in \mathbb{N}} |\varphi_n(\xi)| < \infty$. Since this holds for every $\xi \in H$, the principle of uniform boundedness (sometimes called the Banach-Steinhaus theorem) gives that

$$\sup_{n \in \mathbb{N}} \|\varphi_n\| < \infty.$$

As $\|\varphi_n\| = \|A_n\eta\|$ for every $n \in \mathbb{N}$, this means that $\sup_{n\in\mathbb{N}} \|A_n\eta\| < \infty$. Since this holds for every $\eta \in H$, the principle of uniform boundness applies again, giving that $\sup_{n\in\mathbb{N}} \|A_n\| < \infty$, as desired.

Exercise 33

Let H be a Hilbert space and $\mathcal{N}(H)$ denote the set of all normal operators in $\mathcal{B}(H)$. Show that the map $T \mapsto T^*$ from $\mathcal{N}(H)$ into $\mathcal{B}(H)$ is SOT-SOT continuous.

Solution. Consider a net $\{T_\alpha\}$ in $\mathcal{N}(H)$ converging to $T \in \mathcal{N}(H)$ in the SOT. Let $\xi \in H$. Then

$$\begin{aligned}\|(T_\alpha^* - T^*)\xi\|^2 =& \langle T_\alpha^*\xi - T^*\xi, T_\alpha^*\xi - T^*\xi\rangle \\ =& \|T_\alpha^*\xi\|^2 + \|T^*\xi\|^2 - \langle T_\alpha^*\xi, T^*\xi\rangle - \langle T^*\xi, T_\alpha^*\xi\rangle \\ =& \|T_\alpha\xi\|^2 + \|T\xi\|^2 - \langle \xi, T_\alpha T^*\xi\rangle - \langle T_\alpha T^*\xi, \xi\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_\alpha \|(T_\alpha^* - T^*)\xi\|^2 &= \|T\xi\|^2 + \|T\xi\|^2 - \langle \xi, TT^*\xi\rangle - \langle TT^*\xi, \xi\rangle \\ &= 2\|T\xi\|^2 - 2\|T^*\xi\|^2 = 0\,.\end{aligned}$$

This shows that $\{T_\alpha^*\}$ converges to $T^* \in \mathcal{N}(H)$ in the SOT, and this proves the claim.

(We used above several times that $\|S^*\xi\| = \|S\xi\|$ for all $\xi \in H$ whenever $S \in \mathcal{N}(H)$, a fact which should be well-known, and is easy to prove.)