MAT4460 - Fall 2026 - Week 43 - Solutions

Exercise 34

Let H be a Hilbert space and $\{T_\alpha\}_{\alpha\in\Lambda}$ be an increasing net of self-adjoint operators in $\mathcal{B}(H)$.

a) Assume that $\sup_{\alpha\in\Lambda}\|T_\alpha\|<\infty$. *Then* $\{T_\alpha:\alpha\in\Lambda\}$ *has a least upper bound* T *in* $\mathcal{B}(H)_{\rm sa}$, *which satisfies that* $T=\lim_\alpha T_\alpha$ *(in the SOT)*:

Solution. Set $M:=\sup_{\alpha\in\Lambda}\|T_\alpha\|\geq 0$. Let $\xi\in H$ and consider the net $\{\langle T_\alpha\xi,\xi\rangle\}_{\alpha\in\Lambda}$ in $\mathbb{R}$. It is increasing since $\{T_\alpha\}_{\alpha\in\Lambda}$ is increasing. Moreover, we have

$$\big|\langle T_\alpha\xi,\xi\rangle\big|\leq\|T_\alpha\|\,\|\xi\|^2\leq M\,\|\xi\|^2$$

for all $\alpha\in\Lambda$. By the completeness axiom for $\mathbb{R}$, we get that the set $\{\langle T_\alpha\xi,\xi\rangle:\alpha\in\Lambda\}$ has a least upper bound $M_\xi\in\mathbb{R}$ satisfying $M_\xi\leq M\,\|\xi\|^2$. We then have $\lim_\alpha\langle T_\alpha\xi,\xi\rangle=M_\xi$. Indeed, given $\varepsilon>0$, we can find $\alpha\in\Lambda$ such that $M_\xi-\varepsilon<\langle T_\alpha\xi,\xi\rangle\leq M_\xi$. For all $\beta\in\Lambda$ such that $\alpha\lesssim\beta$, we then have $M_\xi-\varepsilon<\langle T_\alpha\xi,\xi\rangle\leq\langle T_\beta\xi,\xi\rangle\leq M_\xi$, hence $|\langle T_\beta\xi,\xi\rangle-M_\xi|<\varepsilon$, and the claim follows. Consider now $\xi,\eta\in H$. Then we get

$$\lim_\alpha\langle T_\alpha\xi,\eta\rangle=\lim_\alpha\,\frac{1}{4}\sum_{k=0}^{3}i^k\langle T_\alpha(\xi+i^k\eta),\xi+i^k\eta\rangle=\frac{1}{4}\sum_{k=0}^{3}i^k\,M_{\xi+i^k\eta}.$$

Setting $B(\xi,\eta):=\frac{1}{4}\sum_{k=0}^{3}i^k\,M_{\xi+i^k\eta}$, we get readily that $B:H\times H\to\mathbb{C}$ is a sesquilinear form such that

$$|B(\xi,\eta)|=\lim_\alpha\big|\langle T_\alpha\xi,\eta\rangle\big|\;\leq\;M\,\|\xi\|\,\|\eta\|$$

for all $\xi,\eta\in H$. By the Riesz representation theorem for bounded sesquilinear forms on H, we get that there exists $T\in\mathcal{B}(H)$ such that $B(\xi,\eta)=\langle T_\alpha\xi,\eta\rangle$, hence $\lim_\alpha\langle T_\alpha\xi,\eta\rangle=\langle T\xi,\eta\rangle$ for all $\xi,\eta\in H$, that is, $T_\alpha\to_\alpha T$ (WOT). Since $T_\alpha^*=T_\alpha$, we get that $T^*=T$. Moreover, we have

$$\langle T_\alpha\xi,\xi\rangle\leq M_\xi=B(\xi,\xi)=\langle T\xi,\xi\rangle$$

for all $\xi\in H$, so T is an upper bound for $\{T_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{B}(H)_{\rm sa}$. In fact, it is the least upper bound:

Indeed, assume $S\in\mathcal{B}(H)_{\rm sa}$ is such that $T_\alpha\leq S$ for all $\alpha\in\Lambda$. Let $\xi\in H$. Then we have $\langle T_\alpha\xi,\xi\rangle\leq\langle S\xi,\xi\rangle$, so

$$\langle T\xi,\xi\rangle=\;\lim_\alpha\,\langle T_\alpha\xi,\xi\rangle\leq\langle S\xi,\xi\rangle.$$

Thus, $T\leq S$.

Finally, we have $\lim_\alpha T_\alpha=T$ (SOT):
Let $\xi\in H$ and $\alpha\in\Lambda$. Then we have $T-T_\alpha\geq 0$, so

$$\begin{aligned}\|(T-T_\alpha)\xi\|^2&=\|(T-T_\alpha)^{1/2}(T-T_\alpha)^{1/2}\xi\|^2\\&\leq\|(T-T_\alpha)^{1/2}\|^2\;\|(T-T_\alpha)^{1/2}\xi\|^2\\&=\|(T-T_\alpha)\|\;\langle(T-T_\alpha)\xi,\xi\rangle\\&\leq\;2M\,\langle(T-T_\alpha)\xi,\xi\rangle.\end{aligned}$$

Since $\langle(T-T_\alpha)\xi,\xi\rangle\to_\alpha 0$, this implies that $\lim_\alpha\|(T-T_\alpha)\xi\|=0$, as desired.

b) *The conclusion in a) also holds if we assume that* $\{T_\alpha : \alpha \in \Lambda\}$ *has an upper bound in* $\mathcal{B}(H)_{\text{sa}}$ (This result is called Vigier's theorem):

Solution. Assume $\{T_\alpha : \alpha \in \Lambda\}$ has an upper bound S in $\mathcal{B}(H)_{\text{sa}}$. Pick $\alpha_0 \in \Lambda$ and consider $\Omega := \{\alpha \in \Lambda : \alpha_0 \lesssim \alpha\}$ as a directed set. For every $\alpha \in \Omega$, we have $0 \leq T_\alpha - T_{\alpha_0} \leq S - T_{\alpha_0}$, hence $\|T_\alpha - T_{\alpha_0}\| \leq \|S - T_{\alpha_0}\|$, and this gives that

$$\|T_\alpha\| \leq \|T_{\alpha_0}\| + \|T_\alpha - T_{\alpha_0}\| \leq \|T_{\alpha_0}\| + \|S - T_{\alpha_0}\|$$

Thus, $\sup_{\alpha\in\Omega} \|T_\alpha\| < \infty$, so a) gives that the set $\{T_\alpha : \alpha \in \Omega\}$ has a least upper bound T in $\mathcal{B}(H)_{\text{sa}}$, which satisfies that $T = \lim_\alpha T_\alpha$ (in the SOT), where the limit is taken over α's belonging to Ω.

Now, if $\lambda \in \Lambda$, then there exists $\mu \in \Lambda$ such that $\lambda \lesssim \mu$ and $\alpha_0 \lesssim \mu$ (since Λ is directed). Thus we get that $\mu \in \Omega$, hence that $T_\lambda \leq T_\mu \leq T$. This shows that T is an upper bound for $\{T_\lambda : \lambda \in \Lambda\}$. Moreover, if T' is any upper bound for $\{T_\lambda : \lambda \in \Lambda\}$, then T' is also an upper bound for $\{T_\alpha : \alpha \in \Omega\}$, so we have $T \leq T'$. Hence, T is the least upper bound for $\{T_\lambda : \lambda \in \Lambda\}$.

Finally, let $\xi \in H$ and $\varepsilon > 0$. Then we can find $\beta_0 \in \Omega$ such that $\|(T_\alpha - T)\xi\| < \varepsilon$ for all $\alpha \in \Omega, \beta_0 \lesssim \alpha$. Consider now any $\lambda \in \Lambda$ such that $\beta_0 \lesssim \lambda$. Since $\alpha_0 \lesssim \beta_0$, we also have $\alpha_0 \lesssim \lambda$, i.e., $\lambda \in \Omega$. Hence we get that $\|(T_\lambda - T)\xi\| < \varepsilon$. This shows that $T = \lim_\lambda T_\lambda$ (in the SOT), where the limit is taken over λ's belonging to Λ, as desired.

Exercise 35

Let H be a (nonzero) Hilbert space.

a) For $\xi, \eta \in H$, set $D_{(\xi,\eta)} := \{\lambda \in \mathbb{C} : |\lambda| \leq \|\xi\|\,\|\eta\|\}$. *Then* $\langle T\xi, \eta\rangle \in D_{(\xi,\eta)}$ *for every* $T \in \mathcal{B}(H)_1$:

Solution. Let $T \in \mathcal{B}(H), \|T\| \leq 1$. Then the Cauchy-Schwarz inequality gives

$$|\langle T\xi, \eta\rangle| \leq \|T\xi\|\|\eta\| \leq \|T\|\|\xi\|\|\eta\| \leq \|\xi\|\|\eta\|,$$

as desired.

b) Equip each $D_{(\xi,\eta)}$ with the relative topology it inherits from the standard topology on $\mathbb{C}$, and equip the product space $\Lambda := \prod_{(\xi,\eta)\in H\times H} D_{(\xi,\eta)}$ with the product topology. Consider the map $\phi : \mathcal{B}(H)_1 \to \Lambda$ given by

$$\phi(T) = \big\{\langle T\xi, \eta\rangle\big\}_{(\xi,\eta)\in H\times H}$$

for each $T \in \mathcal{B}(H)_1$. *Then the range* Ω *of* ϕ *is compact*:

Solution. Since every disk $D_{(\xi,\eta)}$ is compact, Tychonov's theorem gives that Λ is compact in the product topology. So to show that Ω is compact, it suffices to check that it is closed in Λ. Let $\{\phi(T_\alpha)\}_{\alpha\in A}$ be a net in Ω converging to some $\{\lambda_{(\xi,\eta)}\}_{(\xi,\eta)\in H\times H}$ in Λ, i.e., satisfying

$$\lim_\alpha \langle T_\alpha\xi, \eta\rangle = \lambda_{(\xi,\eta)}$$

for every $(\xi, \eta) \in H \times H$. Since $\|T_\alpha\| \leq 1$ for each $\alpha \in A$, we have that

$$|\langle T_\alpha\xi, \eta\rangle| \leq \|T_\alpha\|\|\xi\|\|\eta\| \leq \|\xi\|\|\eta\|$$

for all $\alpha \in A$ and all $(\xi, \eta) \in H \times H$. This implies that $|\lambda_{(\xi,\eta)}| \leq \|\xi\|\|\eta\|$ for all $(\xi, \eta) \in H \times H$. Thus we readily get that the map $(\xi, \eta) \mapsto \lambda_{(\xi,\eta)}$ is a bounded sesquilinear form on $H \times H$

having norm ≤ 1. The Riesz representation theorem gives then that there exists $T \in \mathcal{B}(H)_1$ such that $\lambda_{(\xi,\eta)} = \langle T\xi, \eta\rangle$ for all $(\xi, \eta) \in H \times H$, i.e., such that $\{\lambda_{(\xi,\eta)}\}_{(\xi,\eta)\in H\times H} = \phi(T)$. Hence, the net $\{\phi(T_\alpha)\}_{\alpha\in A}$ converges to $\phi(T) \in \Omega$. This shows that Ω is closed in Λ, as desired.

c) Consider now ϕ as a map from $\mathcal{B}(H)_1$ onto Ω, and equip $\mathcal{B}(H)_1$ with the relative WOT-topology. *Then ϕ is a homeomorphism, so it follows that $\mathcal{B}(H)_1$ is WOT-compact*:

Solution. The map $\phi : \mathcal{B}(H)_1 \to \Omega$ is trivially surjective. It also injective since a bounded operator is determined by its associated sesquilinear form. So ϕ is bijection. If $\{T_\alpha\}_{\alpha\in A}$ is a net in $\mathcal{B}(H)_1$ and $T \in \mathcal{B}(H)_1$, it is obvious from the solution of b) that $\{\phi(T_\alpha)\}_{\alpha\in A}$ converges to T in the relative WOT-topology if and only if $\{\phi(T_\alpha)\}_{\alpha\in A}$ converges to $\phi(T)$ in the product topology. Thus, ϕ and ϕ^{-1} are continuous, i.e., ϕ is a homeomorphism. Since Ω is compact by b), we get that $\mathcal{B}(H)_1 = \phi^{-1}(\Omega)$ is compact in the relative WOT-topology.

d) Let $\mathcal{M}$ be a von Neumann algebra on H. *Then the unit ball $\mathcal{M}_1$ is WOT-compact*:

Solution. Since $\mathcal{M}_1 = \mathcal{M} \cap \mathcal{B}(H)_1$ and $\mathcal{M}$ is WOT-closed, this follows from b).

Exercise 36

Let H be a Hilbert space.

a) *$\mathcal{B}(H)_1$ is SOT-closed*:

Solution. This can be deduced from c) in Exercise 35. But here is a direct argument: Assume $\{T_\alpha\}$ is a net in $\mathcal{B}(H)_1$ converging in the SOT to some $T \in \mathcal{B}(H)$. Let $\xi \in H, \|\xi\| \leq 1$. Then $\|T_\alpha\xi\| \leq \|T_\alpha\|\,\|\xi\| \leq 1$ for all α, so $\|T\xi\| = \lim_\alpha \|T_\alpha\xi\| \leq 1$. Thus, $\|T\| \leq 1$, i.e., $T \in \mathcal{B}(H)_1$.

Hence, if $\mathcal{B}$ is a SOT-closed $*$-subalgebra of $\mathcal{B}(H)$, then $\mathcal{B}_{1,\mathrm{sa}} := \mathcal{B} \cap \mathcal{B}(H)_1 \cap \mathcal{B}(H)_{\mathrm{sa}}$ is SOT-closed (being the intersection of SOT-closed sets).

b) Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(H)$. Set $\mathcal{B} = \overline{\mathcal{A}}^{\mathrm{SOT}}$ and $\mathcal{C} = \overline{\mathcal{A}}^{\|\cdot\|}$, so $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{B}$. *Then we have*:

$$(1)\quad \overline{\mathcal{A}_{1,\mathrm{sa}}}^{\mathrm{SOT}} = \overline{\mathcal{C}_{1,\mathrm{sa}}}^{\mathrm{SOT}}, \text{and} \quad (2)\quad \left(\overline{\mathcal{C}}^{\mathrm{SOT}}\right)_{1,\mathrm{sa}} = \mathcal{B}_{1,\mathrm{sa}}\,.$$

Solution. (1): Since $\mathcal{A}_{1,\mathrm{sa}} \subseteq \mathcal{C}_{1,\mathrm{sa}}$, we get $\overline{\mathcal{A}_{1,\mathrm{sa}}}^{\mathrm{SOT}} \subseteq \overline{\mathcal{C}_{1,\mathrm{sa}}}^{\mathrm{SOT}}$.
To show the reverse inclusion, it suffices to show that $\mathcal{C}_{1,\mathrm{sa}} \subseteq \overline{\mathcal{A}_{1,\mathrm{sa}}}^{\mathrm{SOT}}$. Let $C \in \mathcal{C}_{1,\mathrm{sa}}$. Pick a sequence $\{A_n\}$ in $\mathcal{A}$ such that $\|C - A_n\| \to_n 0$. We then have $\lim_{n\to\infty} \|A_n\| = \|C\| \leq 1$. If $\|C\| < 1$, we can choose $N \in \mathbb{N}$ such that $\|A_n \leq 1$ for all $n \geq N$. On the other hand, if $\|C\| = 1$, we can choose $M \in \mathbb{N}$ such that $\|A_n\| > 0$ for all $n \geq M$. It is then clear that the sequence $\{\frac{1}{\|A_n\|}A_n\}_{n\geq M}$ converges in norm to C. Thus, by relabelling if necessary, we can always assume that $\|A_n\| \leq 1$ for all $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, set $B_n := (A_n + A_n^*)/2$. Then each B_n belongs to $\mathcal{A}_{1,\mathrm{sa}}$. Further, since the involution is norm-continuous and C is self-adjoint, $\{B_n\}$ converges in norm to $(C + C^*)/2 = C$. In particular, $\{B_n\}$ converges in the SOT to C. Thus, $C \in \overline{\mathcal{A}_{1,\mathrm{sa}}}^{\mathrm{SOT}}$. This shows that $\mathcal{C}_{1,\mathrm{sa}} \subseteq \overline{\mathcal{A}_{1,\mathrm{sa}}}^{\mathrm{SOT}}$, as desired.

(2): It clearly suffices to show that $\overline{\mathcal{C}}^{\mathrm{SOT}} = \mathcal{B}$. Since $\mathcal{A} \subseteq \mathcal{C}$, we get $\mathcal{B} = \overline{\mathcal{A}}^{\mathrm{SOT}} \subseteq \overline{\mathcal{C}}^{\mathrm{SOT}}$.
On the other hand,

$$\mathcal{C} = \overline{\mathcal{A}}^{\|\cdot\|} \subseteq \overline{\mathcal{A}}^{\mathrm{SOT}} = \mathcal{B},$$

so $\overline{\mathcal{C}}^{\mathrm{SOT}} \subseteq \mathcal{B}$. Thus, $\overline{\mathcal{C}}^{\mathrm{SOT}} = \mathcal{B}$.

Exercise 37.

Let $\mathcal{A}$ be a C^*-algebra and π be a nonzero representation of $\mathcal{A}$ on a Hilbert space H. *Then π is irreducible if and only if* $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}} = \mathcal{B}(H)$:

Solution. Assume π is irreducible. Then $\pi(\mathcal{A})$ is a nondegenerate $*$-subalgebra of $\mathcal{B}(H)$, and we have seen that $\pi(\mathcal{A})'' = \mathcal{B}(H)$. As $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}} = \pi(\mathcal{A})''$ by von Neumann's double commutant theorem, we get that $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}} = \mathcal{B}(H)$.

Conversely, assume $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}} = \mathcal{B}(H)$. Since $\pi(\mathcal{A})''$ is SOT-closed and $\pi(A) \subseteq \pi(\mathcal{A})''$, we get

$$\mathcal{B}(H) = \overline{\pi(\mathcal{A})}^{\mathrm{SOT}} \subseteq \pi(\mathcal{A})'' \subseteq \mathcal{B}(H)\,,$$

hence $\pi(\mathcal{A})'' = \mathcal{B}(H)$, which is equivalent to π being irreducible.

Exercise 38.

Let $\mathcal{A}$ be a C^*-algebra and let π, ρ be two unitarily equivalent representations of $\mathcal{A}$ (i.e., $\pi \simeq \rho$). *Then π is irreducible if and only if ρ is irreducible*:

Solution. By symmetry, it suffices to the forward implication. Assume that π is nonzero and irreducible. Let $U : H \to K$ be a unitary operator such that $\rho(A) = U\pi(A)U^*$ for all $A \in \mathcal{A}$, where H (resp. K) denotes the Hilbert space on which π (resp. ρ) acts.
Let N be a closed subspace of K which invariant under ρ. Then $M := U^*(N)$ is a closed subspace of H. Let $\xi \in M$, so $\xi = U^*\eta$ for some $\eta \in N$. Then for each $A \in \mathcal{A}$ we have that $\rho(A)\eta \in N$, hence that

$$\pi(A)\xi = U^*\rho(A)UU^*\eta = U^*\rho(A)\eta \in U^*(N) = M.$$

Thus, M is invariant under π, so M is equal to either $\{0\}$ or H. This implies that $N = U(M)$ is equal to $\{0\}$ or K. It follows that ρ is irreducible.

Exercise 39.

Let $\mathcal{A}$ be an abelian C^*-algebra and recall that $\widehat{\mathcal{A}}$ denotes the set of all characters of $\mathcal{A}$ (i.e., all nonzero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$). We identify $\mathbb{C}$ (as a C^*-algebra) with $\mathcal{B}(\mathbb{C})$ via the $*$-isomorphism $\lambda \mapsto \ell_\lambda$, where $\ell_\lambda(\gamma) = \lambda\gamma$ for $\lambda, \gamma \in \mathbb{C}$.

a) Let $\phi \in \widehat{\mathcal{A}}$ and define $L_\phi : \mathcal{A} \to \mathcal{B}(\mathbb{C})$ by $L_\phi(A) = \ell_{\phi(A)}$ for all $\mathcal{A} \in \mathcal{A}$. *Then L_ϕ is a nonzero irreducible representation of $\mathcal{A}$ on $\mathbb{C}$*:

Solution. It is straightforward to verify that L_ϕ is a $*$-homomorphism into $\mathcal{B}(\mathbb{C})$. As the only subspaces of $\mathbb{C}$ are $\{0\}$ and $\mathbb{C}$, L_ϕ is automatically irreducible. Since ϕ is nonzero, $\phi(A) \neq 0$ for some $A \in \mathcal{A}$, and then $L_\phi(A) = L_{\phi(A)} \neq 0$. So L_ϕ is nonzero.

b) Conversely, let π be a nonzero irreducible representation of $\mathcal{A}$. *Then π is unitarily equivalent to L_ϕ for some $\phi \in \widehat{\mathcal{A}}$*:

Solution. Using Exercise 37 we get that $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}} = \mathcal{B}(H)$. As $\mathcal{A}$ is abelian, $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}}$ is also abelian. Hence $\mathcal{B}(H)$ is abelian, so $\dim(H) = 1$. (It can not be equal to zero since π is nonzero.) Thus $H = \mathrm{span}\{\xi\}$ for some unit vector $\xi \in H$. For every $A \in \mathcal{A}$ we get that $\pi(A) = \phi(A)\xi$ for

some unique $\phi(A) \in \mathbb{C}$. One checks readily that the map $\phi : A \mapsto \phi(A)$ is a character on $\mathcal{A}$. Now the map $U : \mathbb{C} \to H$ given by $U(\lambda) = \lambda\xi$ for each $\lambda \in \mathbb{C}$ is clearly unitary, and we have

$$UL_\phi(A)U^* = \pi(A) \text{ for all } A \in \mathcal{A},$$

which shows the assertion.

The map $\phi \mapsto [L_\phi]$ from $\widehat{\mathcal{A}}$ to $\{[\pi] : \pi$ is a nonzero irreducible representation of $\mathcal{A}\}$ is a bijection:

Solution. Part a) shows that the map is well-defined, and part b) that it is surjective. Assume $[L_{\phi_1}] = [L_{\phi_2}]$. Since L_{ϕ_1} and L_{ϕ_2} both act on $\mathbb{C}$, it is clear that this implies that $L_{\phi_1} = L_{\phi_2}$. Then for each $A \in \mathcal{A}$, we get $\phi_1(A) = [L_{\phi_1}(A)](1) = L_{\phi_2}(A)](1) = \phi_2(A)$. Thus, $\phi_1 = \phi_2$. This shows that the map is injective.

c) Let again $\phi \in \widehat{\mathcal{A}}$ and consider ϕ as a state on $\mathcal{A}$. *Then the associated GNS-representation π_ϕ is unitarily equivalent to L_ϕ*:

Solution. Since ϕ is a $*$-homomorphism, we have that

$$\mathcal{N} := \{A \in \mathcal{A} \mid \phi(A^*A) = 0\} = \{A \in \mathcal{A} \mid \phi(A) = 0\} = \mathrm{Ker}(\phi).$$

Thus we get that $\mathcal{A}/\mathcal{N} = \mathcal{A}/\mathrm{Ker}(\phi) \simeq \mathbb{C}$ (via the $*$-isomorphism given by $A + \mathcal{N} \mapsto \phi(A)$). This gives that the GNS-Hilbert space associated to ϕ is $H_\phi = \mathcal{A}/\mathcal{N} \simeq \mathbb{C}$, with the inner product given by

$$\langle A + \mathcal{N}, B + \mathcal{N} \rangle = \phi(B^*A) = \phi(A)\overline{\phi(B)}.$$

It is then almost obvious that if we consider $\mathbb{C}$ as a Hilbert space equipped with its standard inner product, then the linear map $V : \mathcal{A}/\mathcal{N} \to \mathbb{C}$ defined by $V(A + \mathcal{N}) = \phi(A)$ is a unitary operator. Moreover, since the GNS-representation $\pi_\phi : \mathcal{A} \to \mathcal{B}(H_\phi)$ is given by $[\pi_\phi(A)](B + \mathcal{N}) = AB + \mathcal{N}$, we get that

$$\begin{aligned}[V\pi_\phi(A)](B + \mathcal{N}) &= V(AB + \mathcal{N}) = \phi(AB) = \phi(A)\phi(B) \\ &= [L_\phi(A](\phi(B)) = [L_\phi(A)](V(B + \mathcal{N})) \\ &= [L_\phi(A)V](B + \mathcal{N})\end{aligned}$$

for all $A, B \in \mathcal{A}$. This implies that $V\pi_\phi(A) = L_\phi(A)V$, i.e., $L_\phi(A) = V\pi_\phi(A)V^*$ for all $A \in \mathcal{A}$. This shows that $L_\phi \simeq \pi_\phi$.

It follows that π_ϕ is irreducible and $[\pi_\phi] = [L_\phi]$:

Solution. Using b) we get that $\pi_\phi \simeq L_\phi$ is irreducible. Moreover, $[\pi_\phi] = [L_\phi]$ since $\pi_\phi \simeq L_\phi$.

Exercise 40.

Let $\mathcal{A}$ be a C^*-algebra. Recall that $\widehat{\mathcal{A}} := \{[\pi] : \pi$ is a nonzero irreducible representation of $\mathcal{A}\}$. Let $\mathcal{J}$ be a (two-sided) closed ideal of $\mathcal{A}$ and π be a nonzero irreducible representation of $\mathcal{A}$ on a Hilbert space H.

a) Assume that $\pi(\mathcal{J}) \neq \{0\}$ and let $\pi_\mathcal{J} : \mathcal{J} \to \mathcal{B}(H)$ denote the representation of $\mathcal{J}$ on H obtained by restriction, i.e., $\pi_\mathcal{J}$ is given by

$$\pi_\mathcal{J}(B) := \pi(B) \quad \text{for all } B \in \mathcal{J}.$$

Then $\pi_{\mathcal{J}}$ is irreducible, and we have that $\pi_{\mathcal{J}} \simeq \rho_{\mathcal{J}}$ if $\pi \simeq \rho$:

Solution. We first show that $\pi_{\mathcal{J}}$ is irreducible. Let ξ be any nonzero vector in H. Since $\pi_{\mathcal{J}}$ is nonzero, it suffices to show that $[\pi_{\mathcal{J}}(\mathcal{J})\xi] = H$, i.e., that $[\pi(\mathcal{J})\xi] = H$. It is clear that $[\pi(\mathcal{J})\xi]$ is invariant under π. Since π is irreducible, we deduce that $[\pi(\mathcal{J})\xi]$ is equal to $\{0\}$ or H.

Assume that $[\pi(\mathcal{J})\xi] = \{0\}$. Then we have that $\xi \in N(\pi(\mathcal{J})) = [\pi(\mathcal{J})H]^{\perp}$, hence that $[\pi(\mathcal{J})H] \neq H$. As $[\pi(\mathcal{J})H]$ is invariant under π, which is irreducible, we must have that $[\pi(\mathcal{J})H] = \{0\}$, i.e., $\pi(\mathcal{J}) = \{0\}$, which gives a contradiction.

Thus we must have that $[\pi(\mathcal{J})\xi] = H$, as desired.

Next, assume that $\pi \simeq \rho$ for some representation ρ of $\mathcal{A}$ on K. Since π is nonzero and irreducible, so is ρ. If $U : H \to K$ is a unitary operator such that $\rho(A) = U\pi(A)U^*$ for all $A \in \mathcal{A}$, then we trivially have that $\rho_{\mathcal{J}}(A) = U\pi_{\mathcal{J}}(A)U^*$ for all $A \in \mathcal{J}$, so $\pi_{\mathcal{J}} \simeq \rho_{\mathcal{J}}$.

b) Assume that $\pi(\mathcal{J}) = \{0\}$. *Then the map $\dot{\pi} : \mathcal{A}/\mathcal{J} \to \mathcal{B}(H)$ given by*

$$\dot{\pi}(A + \mathcal{J}) = \pi(A)$$

for every $A \in \mathcal{A}$ is a well-defined nonzero irreducible representation of $\mathcal{A}/\mathcal{J}$ on H:

Solution. Assume that $A, B \in \mathcal{A}$ and $A + \mathcal{J} = B + \mathcal{J}$, so $B = A + J$ for some $J \in \mathcal{J}$. Then we have $\pi(B) = \pi(A) + \pi(J) = \pi(A)$. Hence the map $\dot{\pi}$ is well-defined. (This is presumably known from a previous course.) Since π is nonzero, it is obvious that $\dot{\pi}$ is nonzero too. It is routine to check that $\dot{\pi}$ is a $*$-homomorphism. Moreover, if ξ be any nonzero vector in H, then we have $[\dot{\pi}(\mathcal{A}/\mathcal{J})\xi] \ = [\pi(\mathcal{A})\xi] = H$. Thus $\dot{\pi}$ is an irreducible representation of $\mathcal{A}/\mathcal{J}$ on H.

We also have that $\dot{\pi} \simeq \dot{\rho}$ if $\pi \simeq \rho$:

Solution. If $\pi \simeq \rho$, it follows readily that ρ is also a nonzero irreducible representation of $\mathcal{A}$ such $\rho(\mathcal{J}) = \{0\}$), and that the unitary U which implements the unitary equivalence between π and ρ also works for $\dot{\pi}$ and $\dot{\rho}$.

c) *The map $[\pi] \mapsto [\dot{\pi}]$ is a bijection from $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) = \{0\}\}$ to $\widehat{\mathcal{A}/\mathcal{J}}$:*

Solution. Part b) gives that this map is well-defined. It is (almost) obvious that it is injective. Finally, assume Φ is a nonzero irreducible representation of $\mathcal{A}/\mathcal{J}$. Then let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be defined by $\pi(A) := \Phi(A + \mathcal{J})$ for all $A \in \mathcal{A}$. Then π is easily seen to be a nonzero irreducible representation of $\mathcal{A}$ satisfying that $\pi(\mathcal{J}) = \{0\}$ and $\dot{\pi} = \Phi$. This shows surjectivity.