MAT4460 - Fall 2026 - Exercises - Week 43 (19.10-23.10)

Exercise 34.

Let H be a Hilbert space and $\{T_\alpha\}_{\alpha\in\Lambda}$ be an increasing net of self-adjoint operators in $\mathcal{B}(H)$.

a) Assume that $\sup_{\alpha\in\Lambda}\|T_\alpha\|<\infty$. Show that $\{T_\alpha:\alpha\in\Lambda\}$ has a least upper bound T in $\mathcal{B}(H)_{\mathrm{sa}}$, which satisfies that $T=\lim_\alpha T_\alpha$ (in the SOT).

b) Show that the conclusion in a) also holds if we assume that $\{T_\alpha:\alpha\in\Lambda\}$ has an upper bound in $\mathcal{B}(H)_{\mathrm{sa}}$. (This result is called Vigier's theorem).

Exercise 35

Let H be a Hilbert space. With some work, it can be deduced from Alaoglu's theorem that the (closed) unit ball $\mathcal{B}(H)_1$ is WOT-compact. A more direct way to prove this (somewhat similar to the one used to prove Alaoglu's theorem) goes as follows.

a) For $\xi,\eta\in H$, set $D_{(\xi,\eta)}:=\{\lambda\in\mathbb{C}:|\lambda|\leq\|\xi\|\,\|\eta\|\}$. Check that $\langle T\xi,\eta\rangle\in D_{(\xi,\eta)}$ for every $T\in\mathcal{B}(H)_1$.

b) Equip each $D_{(\xi,\eta)}$ with the relative topology it inherits from the standard topology on $\mathbb{C}$, and equip the product space $\Lambda:=\prod_{(\xi,\eta)\in H\times H}D_{(\xi,\eta)}$ with the product topology. Consider the map $\phi:\mathcal{B}(H)_1\to\Lambda$ given by

$$\phi(T):=\big\{\langle T\xi,\eta\rangle\big\}_{(\xi,\eta)\in H\times H}$$

for each $T\in\mathcal{B}(H)_1$. Let Ω denote the range of ϕ, i.e., $\Omega=\phi(\mathcal{B}(H)_1)$. Show that Ω is compact.

c) Consider now ϕ as a map from $\mathcal{B}(H)_1$ onto Ω, and equip $\mathcal{B}(H)_1$ with the relative WOT-topology. Show that ϕ is then a homeomorphism. Deduce that $\mathcal{B}(H)_1$ is WOT-compact.

d) Let $\mathcal{M}$ be a von Neumann algebra on H. Show that the unit ball $\mathcal{M}_1$ is WOT-compact.

Exercise 36

Let H be a Hilbert space. We used the following facts in the proof of Kaplansky's density theorem.

a) Check that $\mathcal{B}(H)_1$ is SOT-closed. Deduce that if $\mathcal{B}$ is a SOT-closed $*$-subalgebra of $\mathcal{B}(H)$, then $\mathcal{B}_{1,\,\mathrm{sa}}:=\mathcal{B}\cap\mathcal{B}(H)_1\cap\mathcal{B}(H)_{\mathrm{sa}}$ is SOT-closed.

b) Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(H)$. Set $\mathcal{B}=\overline{\mathcal{A}}^{\mathrm{SOT}}$ and $\mathcal{C}=\overline{\mathcal{A}}^{\|\cdot\|}$, so $\mathcal{A}\subseteq\mathcal{C}\subseteq\mathcal{B}$. Check that

$$\overline{\mathcal{A}_{1,\,\mathrm{sa}}}^{\mathrm{SOT}}=\overline{\mathcal{C}_{1,\,\mathrm{sa}}}^{\mathrm{SOT}}\quad\text{and}\quad\big(\overline{\mathcal{C}}^{\mathrm{SOT}}\big)_{1,\,\mathrm{sa}}=\mathcal{B}_{1,\,\mathrm{sa}}\,.$$

Note. In class, we proved that $\overline{\mathcal{C}_{1,\,\mathrm{sa}}}^{\mathrm{SOT}}=\big(\overline{\mathcal{C}}^{\mathrm{SOT}}\big)_{1,\,\mathrm{sa}}$. Thus, using the above equalities, we get $\overline{\mathcal{A}_{1,\,\mathrm{sa}}}^{\mathrm{SOT}}=\mathcal{B}_{1,\,\mathrm{sa}}$, i.e., the second part of Kaplansky's density theorem holds.

Exercise 37

Let $\mathcal{A}$ be a C^*-algebra and assume π is a nonzero representation of $\mathcal{A}$ on a Hilbert space H. Show that π is irreducible if and only if $\overline{\pi(\mathcal{A})}^{\mathrm{SOT}}=\mathcal{B}(H)$.

Exercise 38.

Let $\mathcal{A}$ be a C^*-algebra and let π, ρ be two unitarily equivalent representations of $\mathcal{A}$ (i.e., $\pi \simeq \rho$). Check that π is irreducible if and only if ρ is irreducible. (Thus it makes sense to say that the equivalence class $[\pi]$ of π w.r.t. unitary equivalence is irreducible if π irreducible.)

Exercise 39.

Let $\mathcal{A}$ be an abelian C^*-algebra and recall that $\widehat{\mathcal{A}}$ denotes the set of all characters of $\mathcal{A}$ (i.e., all nonzero homomorphisms of $\mathcal{A}$ into $\mathbb{C}$). We note that $\mathbb{C}$ can be identified (as a C^*-algebra) with $\mathcal{B}(\mathbb{C})$ via the $*$-isomorphism $\lambda \mapsto \ell_\lambda$, where $\ell_\lambda(\gamma) = \lambda\gamma$ for $\lambda, \gamma \in \mathbb{C}$.

a) Let $\phi \in \widehat{\mathcal{A}}$ and define $L_\phi : \mathcal{A} \to \mathcal{B}(\mathbb{C})$ by $L_\phi(A) = \ell_{\phi(A)}$ for all $\mathcal{A} \in \mathcal{A}$. Check that L_ϕ is a nonzero irreducible representation of $\mathcal{A}$ on $\mathbb{C}$. (It is usual to identify ϕ with L_ϕ, i.e., to think of ϕ as an irreducible representation of $\mathcal{A}$ on $\mathbb{C}$).

b) Conversely, let π be a nonzero irreducible representation of $\mathcal{A}$. Show that π is unitarily equivalent to L_ϕ for some $\phi \in \widehat{\mathcal{A}}$. Then deduce that the map $\phi \mapsto [L_\phi]$ gives a bijection between $\widehat{\mathcal{A}}$ and the set $\{[\pi] : \pi \text{ is a nonzero irreducible representation of } \mathcal{A}\}$.

c) Let again $\phi \in \widehat{\mathcal{A}}$ and consider ϕ as a state on $\mathcal{A}$. Check that the associated GNS-representation π_ϕ is unitarily equivalent to L_ϕ. Deduce that π_ϕ is irreducible and $[\pi_\phi] = [L_\phi]$.

Exercise 40.

Let $\mathcal{A}$ be a C^*-algebra. Recall that $\widehat{\mathcal{A}} := \{[\pi] : \pi \text{ is a nonzero irreducible representation of } \mathcal{A}\}$. Let $\mathcal{J}$ be a (two-sided) closed ideal of $\mathcal{A}$ and π be a nonzero irreducible representation of $\mathcal{A}$ on a Hilbert space H.

a) Assume that $\pi(\mathcal{J}) \neq \{0\}$ and let $\pi_\mathcal{J} : \mathcal{J} \to \mathcal{B}(H)$ denote the representation of $\mathcal{J}$ on H obtained by restriction, i.e., $\pi_\mathcal{J}$ is given by

$$\pi_\mathcal{J}(B) := \pi(B) \quad \text{for all } B \in \mathcal{J}.$$

Show that $\pi_\mathcal{J}$ is irreducible, and that $\pi_\mathcal{J} \simeq \rho_\mathcal{J}$ if $\pi \simeq \rho$.

b) Assume that $\pi(\mathcal{J}) = \{0\}$. Check that the map $\dot{\pi} : \mathcal{A}/\mathcal{J} \to \mathcal{B}(H)$ given by

$$\dot{\pi}(A + \mathcal{J}) = \pi(A)$$

for every $A \in \mathcal{A}$ is a well-defined nonzero irreducible representation of $\mathcal{A}/\mathcal{J}$ on H. Check also that $\dot{\pi} \simeq \dot{\rho}$ if $\pi \simeq \rho$.

c) Show that the map $[\pi] \mapsto [\dot{\pi}]$ is a bijection from $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) = \{0\}\}$ to $\widehat{\mathcal{A}/\mathcal{J}}$.

Comment. It will follow from an exercise in the compulsory assignment that the map $[\pi] \mapsto [\pi_\mathcal{J}]$ is a bijection from $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) \neq \{0\}\}$ to $\widehat{\mathcal{J}}$. Hence, using b), we will then be able to conclude that the set $\widehat{\mathcal{A}}$ may be thought of as the disjoint union of $\widehat{\mathcal{J}}$ and $\widehat{\mathcal{A}/\mathcal{J}}$.