MAT4460 - Fall 2026 - Solutions - Week 44

Exercise 41

Let $\mathcal{A}$ be a commutative C^*-algebra, $\mathcal{A} \neq \{0\}$, and $\varphi \in S(\mathcal{A})$.

Then φ is pure if and only if φ is multiplicative (i.e., φ is a character on $\mathcal{A}$):

Solution. Assume first φ is pure. Then π_φ is irreducible, so, by part b) of Exercise 39, π_φ is unitary equivalent to the representation $L_\varphi : \mathcal{A} \to \mathcal{B}(\mathbb{C})$ given by $L_\varphi(A)\lambda = \varphi(A)\lambda$ for all $A \in \mathcal{A}$. It follows readily that φ is multiplicative.

Conversely, assume φ is multiplicative. Then part c) of Exercise 39 gives that π_φ is irreducible. Hence, φ is pure.

Exercise 42

Let $n \in \mathbb{N}$, $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi \in S(\mathcal{A})$.

Then φ is pure if and only if φ is a vector state on $\mathcal{A}$:

Solution. Since $\varphi \in S(\mathcal{A})$, we know that there exists $S \in \mathcal{A}$ such that $S \geq 0$, $\mathrm{Tr}(S) = 1$ and $\varphi(A) = \mathrm{Tr}(AS)$ for all $A \in \mathcal{A}$. Letting $\{\xi_j\}_{j=1}^n$ be an o.n.b. for $\mathbb{C}^n$ consisting of eigenvectors for S, and $\{\lambda_j\}_{j=1}^n$ be the corresponding listing of eigenvalues, we have $\lambda_j \geq 0$ for $j = 1, \ldots, n$, $\sum_{j=1}^n \lambda_j = 1$, and $S = \sum_{j=1}^n \lambda_j P_j$, where each P_j denote the standard matrix of the orthogonal projection of $\mathbb{C}^n$ onto $\mathrm{Span}(\xi_j)$. Thus, we get that

$$\varphi(A) = \sum_{j=1}^n \lambda_j \,\mathrm{Tr}(AP_j) = \sum_{j=1}^n \lambda_j \,\langle A\xi_j, \xi_j\rangle \quad \text{for all } A \in \mathcal{A}\,.$$

Thus, denoting for each j the vector state ω_j on $\mathcal{A}$ associated to ξ_j, we have $\varphi = \sum_{j=1}^n \lambda_j\, \omega_j$.

Assume first φ is pure. Since $\mathcal{A}$ is unital, we know that φ is an extreme point of $S(\mathcal{A})$. Thus, φ can only be written as a convex combination of elements in $S(\mathcal{A})$ in the trivial way, so we must have $\varphi = \omega_j$ for exactly one $j \in \{1, \ldots, n\}$. In particular, φ is a vector state.

Conversely, assume φ is a vector state, say $\varphi = \omega_\xi$ for a unit vector $\xi \in \mathbb{C}^n$. Then we have

$$\varphi(A) = \langle A\xi, \xi\rangle = \langle \pi_\varphi(A)\xi_\varphi, \xi_\varphi\rangle \quad \text{for all } A \in \mathcal{A}.$$

As ξ is cyclic for the standard representation π of $\mathcal{A}$ on $\mathbb{C}^n$, we get from the uniqueness part of the GNS-construction that $\pi_\varphi \simeq \pi$. As π is irreducible, π_φ is too, cf. Exercise 38. Thus, φ is pure.

Exercise 43

Let $\mathcal{A}$ be a unital C^*-algebra and $\varphi \in S(\mathcal{A})$. Assume $B \in \mathcal{A}$ is invertible and such that $\varphi(B^*B) = 1$ (for instance, B can be a unitary in $\mathcal{A}$). Let $\psi : \mathcal{A} \to \mathbb{C}$ be given by

$$\psi(A) = \varphi(B^*AB) \quad \text{for all } A \in \mathcal{A}\,.$$

Then ψ is a state on $\mathcal{A}$, which is pure whenever φ is pure:

Solution. We get from part a) of Problem 4 in the compulsory assignment that ψ is positive and $\pi_\psi \simeq \pi_\varphi$. Since $\psi(I) = \varphi(B^*B) = 1$, ψ is a state on $\mathcal{A}$. Further, if φ is pure, then π_φ is irreducible, hence π_ψ is too (by Exercise 38), which implies that ψ is pure.

Exercise 44

Let $\mathcal{J}$ be a nonzero closed (two-sided) ideal of a C^*-algebra $\mathcal{A}$ and assume that ρ is a nondegenerate representation of $\mathcal{J}$ on a nonzero Hilbert space H (as in Problem 5 of the compulsory assignment). Let $\widetilde{\rho}$ denote the canonical extension of ρ from $\mathcal{J}$ to $\mathcal{A}$.

a) *$\widetilde{\rho}$ is irreducible whenever ρ is irreducible*:

Solution. Assume ρ is irreducible. Let K be a closed subspace of H which is $\widetilde{\rho}$-invariant. Since $\widetilde{\rho}$ extends ρ, K is then ρ-invariant. Hence, we have $K = \{0\}$ or $K = H$. This shows that $\widetilde{\rho}$ is irreducible.

b) *The map $[\rho] \mapsto [\widetilde{\rho}]$ is a bijection from $\widehat{\mathcal{J}}$ onto $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) \neq \{0\}\}$*:

Solution. If $[\rho] \in \widehat{\mathcal{J}}$, then $\widetilde{\rho}_{|\mathcal{J}} = \rho$ is nonzero on $\mathcal{J}$. Hence, using part a), we get that the assignment $[\rho] \mapsto [\widetilde{\rho}]$ is a well-defined map from $\widehat{\mathcal{J}}$ to $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) \neq \{0\}\}$.

Assume $[\widetilde{\rho}] = [\widetilde{\rho'}]$ for $[\rho], [\rho'] \in \widehat{\mathcal{J}}$. Then $\widetilde{\rho}$ and $\widetilde{\rho'}$ are unitary equivalent. Since $\widetilde{\rho}$ and $\widetilde{\rho'}$ are extensions of ρ and ρ', respectively, it is immediate that ρ and ρ' are unitary equivalent. Hence, $[\rho] = [\rho']$. This shows that the map is injective.

Finally, let $[\pi] \in \widehat{\mathcal{A}}$ be such that $\pi(\mathcal{J}) \neq \{0\}$. Set $\rho := \pi_{|\mathcal{J}}$. Then ρ is a nonzero representation of $\mathcal{J}$, which is irreducible by part a) of Exercise 40. Thus, $[\rho] \in \widehat{\mathcal{J}}$. Since π is an extension of ρ, we get from the uniqueness of the canonical extension (cf. Problem 5 in the compulsory assignment) that $\pi = \widetilde{\rho}$. Thus, $[\pi] = [\widetilde{\rho}]$. This shows surjectivity.

Hence, we have shown that the map $[\rho] \mapsto [\widetilde{\rho}]$ is a bijection.

Combining part b) above with part c) of Exercise 40, we conclude that $\widehat{\mathcal{A}}$ can be identified with the disjoint union of $\widehat{\mathcal{J}}$ and $\widehat{\mathcal{A}/\mathcal{J}}$.