MAT4460 - Fall 2026 - Exercises - Week 44 (26.10-30.10)

Exercise 41

Let $\mathcal{A}$ be a commutative C^*-algebra, $\mathcal{A} \neq \{0\}$, and $\varphi \in S(\mathcal{A})$. Show that φ is pure if and only if φ is multiplicative (i.e., φ is a character on $\mathcal{A}$).

Exercise 42

Let $n \in \mathbb{N}$, $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi \in S(\mathcal{A})$. Show that φ is pure if and only if φ is a vector state on $\mathcal{A}$ (i.e., there exists some unit vector $\xi \in \mathbb{C}^n$ such that $\varphi(A) = \langle A\xi, \xi \rangle$ for all $A \in \mathcal{A}$.

Exercise 43

Let $\mathcal{A}$ be a unital C^*-algebra and $\varphi \in S(\mathcal{A})$. Assume $B \in \mathcal{A}$ is invertible and such that $\varphi(B^*B) = 1$ (for instance, B can be a unitary in $\mathcal{A}$). Let $\psi : \mathcal{A} \to \mathbb{C}$ be given by

$$\psi(A) = \varphi(B^*AB) \quad \text{for all } A \in \mathcal{A}\,.$$

Show that ψ is a state on $\mathcal{A}$, which is pure whenever φ is pure.

Exercise 44

Let $\mathcal{J}$ be a nonzero closed (two-sided) ideal of a C^*-algebra $\mathcal{A}$ and assume that ρ is a nondegenerate representation of $\mathcal{J}$ on a nonzero Hilbert space H (as in Problem 5 of the compulsory assignment). Let $\widetilde{\rho}$ denote the canonical extension of ρ from $\mathcal{J}$ to $\mathcal{A}$.

a) Show that $\widetilde{\rho}$ is irreducible whenever ρ is irreducible.

b) Explain why the map $[\rho] \mapsto [\widetilde{\rho}]$ is a bijection from $\widehat{\mathcal{J}}$ onto $\{[\pi] \in \widehat{\mathcal{A}} : \pi(\mathcal{J}) \neq \{0\}\}$. [1]

Combining the result above with part c) of Exercise 40, we can now conclude that $\widehat{\mathcal{A}}$ can be thought of as the disjoint union of $\widehat{\mathcal{J}}$ and $\widehat{\mathcal{A}/\mathcal{J}}$.

[1] We recall that if $\mathcal{A}$ is a nonzero C^* algebra, then $\widehat{\mathcal{A}} = \{[\pi] : \pi \text{ is a nonzero irreducible representation of } \mathcal{A}\}$, where $[\pi] = \{\sigma : \sigma \text{ is a representation of } \mathcal{A} \text{ whch is unitarily equivalent to } \pi\}$.