MAT4460 - Fall 2026 - Exercises - Week 45 (03.11-07.11)

Exercise 45

Let H be a nonzero Hilbert space, and let $\mathcal{M}$ be a $*$-subalgebra of $\mathcal{B}(H)$ containing the identity operator I_H. Recall that $\mathcal{M}_1$ denotes the unit ball of $\mathcal{M}$. Show that the following conditions are equivalent:

(i) $\mathcal{M}$ is a von Neumann algebra.

(ii) $\mathcal{M}_1$ is WOT-closed.

(iii) $\mathcal{M}_1$ is SOT-closed.

Comment: In view of a previous exercise, these conditions are also equivalent to $\mathcal{M}_1$ being WOT-compact.

Exercise 46

Let H, K be nonzero Hilbert spaces, and consider a map $\alpha : \mathcal{B}(H) \to \mathcal{B}(K)$.

Show that α is a $*$-isomorphism if and only if there exists a unitary operator $U : H \to K$ such that

$$\alpha(T) = UTU^* \quad \text{for all } T \in \mathcal{B}(H).$$

Hint: Use that a similar result holds for a $*$-isomorphism $\mathcal{K}(H) \to \mathcal{K}(K)$.

Comment. This shows in particular that every $*$-automorphism of $\mathcal{B}(H)$ is *inner*, i.e., is associated to a unitary operator $U : H \to H$ as above.

Exercise 47.

a) Assume V is a (complex) finite-dimensional vector space. Show that there exists a unique topology making V into a locally convex Hausdorff topological vector space.

b) Use a) to prove that a nonzero finite-dimensional C^*-algebra $\mathcal{A}$ is unital.

Hint. Consider the universal representation of $\mathcal{A}$ and an approximate unit for $\mathcal{A}$.