

MAT4460 – Fall 2026

Mandatory assignment 1 of 1

Submission deadline

Thursday, October 22, 2026, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with $\text{\LaTeX}$). The assignment must be submitted as a single PDF file. Scanned pages should be clearly legible. The submission must contain your name and the course number.

You should strive to give a clear presentation with all necessary explanations. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject.

To get a passing grade, you only have to hand in your assignment with your name and the course number. The main aim with this assignment is to give you the opportunity to get some feedback on your work.

Application for postponed delivery

If you would like to apply for a postponement of the submission deadline due to illness or other reasons, you have to send an e-mail to the lecturer [Erik Bédos](#) no later than the same day as the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1. Let $\{\mathcal{A}_j\}_{j \in J}$ denote a family of C^* -algebras indexed by a nonempty set J . Set

$$\mathcal{A} = \left\{ \{A_j\} \in \prod_{j \in J} \mathcal{A}_j : \sup_{j \in J} \|A_j\| < \infty \right\}.$$

Define natural pointwise operations on $\mathcal{A}$ by

$$\{A_j\} + \{B_j\} = \{A_j + B_j\}, \quad \lambda \{A_j\} = \{\lambda A_j\},$$

$$\{A_j\} \cdot \{B_j\} = \{A_j B_j\}, \quad \text{and} \quad \{A_j\}^* = \{(A_j)^*\}$$

for all $\{A_j\}, \{B_j\} \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. You can take as granted that these operations are meaningful and make $\mathcal{A}$ into a $*$ -algebra.

Moreover, for $\{A_j\} \in \mathcal{A}$, set

$$\|\{A_j\}\| := \sup_{j \in J} \|A_j\|.$$

It is straightforward to check that this gives a submultiplicative norm on $\mathcal{A}$ (and you are not asked to do this).

a) Show that $\mathcal{A}$ becomes a C^* -algebra.

Comment: It is common to denote $\mathcal{A}$ by $\bigoplus_{j \in J} \mathcal{A}_j$ and call it the *direct sum* of the family $\{\mathcal{A}_j\}_{j \in J}$. When $J = \{1, 2, \dots, n\}$, it is usual to denote $\mathcal{A}$ by $\mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_n$. Also, if $\mathcal{A}_j = \mathcal{B}$ for all $j \in J$, one writes $\mathcal{A} = \ell^\infty(J, \mathcal{B})$.

b) Set $\mathcal{A}_0 = \left\{ \{A_j\} \in \mathcal{A} \mid \text{the function } j \mapsto \|A_j\| \text{ belongs to } c_0(J) \right\}$.

In other words, $\{A_j\} \in \mathcal{A}$ belongs to $\mathcal{A}_0$ means that for any given $\varepsilon > 0$, there exists a finite subset F of J such that $\|A_j\| < \varepsilon$ for all $j \in J \setminus F$.

Show that $\mathcal{A}_0$ is a closed (two-sided) ideal of $\mathcal{A}$.

Problem 2. Let $\mathcal{A}$ be a unital C^* -algebra and φ be a state on $\mathcal{A}$.

a) Let $A \in \mathcal{A}$. Consider the following conditions:

$$(i) \quad |\varphi(A)|^2 = \varphi(A^* A).$$

$$(ii) \quad \varphi(BA) = \varphi(A)\varphi(B) \text{ for all } B \in \mathcal{A}.$$

$$(iii) \quad |\varphi(A)|^2 = \varphi(AA^*).$$

$$(iv) \quad \varphi(AB) = \varphi(A)\varphi(B) \text{ for all } B \in \mathcal{A}.$$

Show that (i) $\Leftrightarrow$ (ii), and (iii) $\Leftrightarrow$ (iv).

Hint: Use the Cauchy-Schwarz inequality in a suitable way.

Comment: It follows that $|\varphi(A)|^2 = \varphi(A^*A) = \varphi(AA^*)$ if and only if $\varphi(AB) = \varphi(BA) = \varphi(A)\varphi(B)$ for all $B \in \mathcal{A}$.

b) Show that φ is multiplicative on $\mathcal{A}$ (so φ is a character on $\mathcal{A}$) if and only if $|\varphi(U)| = 1$ for every unitary $U \in \mathcal{A}$.

Problem 3. Let φ be a positive linear functional on a C^* -algebra $\mathcal{A}$. We recall that φ is called faithful when $\{A \in \mathcal{A} : \varphi(A^*A) = 0\} = \{0\}$.

a) Show that the GNS-representation $\pi_\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_\varphi)$ is faithful whenever φ is faithful.

b) Assume that $\mathcal{A} = \mathcal{B}(H)$ for some Hilbert space H having dimension at least 2. Let $\xi \in H \setminus \{0\}$ and φ be the positive linear functional on $\mathcal{A}$ given by

$$\varphi(A) = \langle A\xi, \xi \rangle \quad \text{for all } A \in \mathcal{B}(H).$$

Show that φ is not faithful, while π_φ is faithful.

Problem 4. a) Let φ be a positive linear functional on a unital C^* -algebra $\mathcal{A}$ and let $B \in \mathcal{A}$ be *invertible*. Define $\psi : \mathcal{A} \rightarrow \mathbb{C}$ by

$$\psi(A) = \varphi(B^*AB) \quad \text{for all } A \in \mathcal{A}.$$

Check that ψ is also a positive linear functional on $\mathcal{A}$. Then show that π_ψ is unitarily equivalent to π_φ (i.e., $\pi_\psi \simeq \pi_\varphi$).

b) Consider $\mathcal{A} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, and let ψ be any *faithful* positive linear functional on $\mathcal{A}$. Show that π_ψ is unitarily equivalent to π_{Tr} (where Tr denotes the canonical trace functional on $\mathcal{A}$).

Problem 5. Let $\mathcal{J}$ be a nonzero closed (two-sided) ideal of a C^* -algebra $\mathcal{A}$. Assume that ρ is a non-degenerate representation of $\mathcal{J}$ on a nonzero Hilbert space H .

a) Let $A \in \mathcal{A}$. Show that there exists a unique operator $\tilde{\rho}(A) \in \mathcal{B}(H)$ satisfying that

$$\tilde{\rho}(A)\rho(B) = \rho(AB) \quad \text{for all } B \in \mathcal{J}.$$

Hint: For the existence part, consider first the case where ρ is cyclic with cyclic vector ξ_0 . Use then an approximate unit for $\mathcal{J}$ in a suitable way to show that $\|\rho(AC)\xi_0\| \leq \|A\| \|\rho(C)\xi_0\|$ for all $C \in \mathcal{J}$.

b) Use part a) to show that there exists a unique representation $\tilde{\rho}$ of $\mathcal{A}$ on H which extends ρ . (You can skip showing linearity and multiplicativity of $\tilde{\rho}$, but not that it preserves adjoints.)

Comment: It is immediate that $\tilde{\rho}$ is non-degenerate. It is called the *canonical extension* of ρ from $\mathcal{J}$ to $\mathcal{A}$.

c) Let σ be a representation of $\mathcal{J}$ on some Hilbert space K , and assume that σ is unitarily equivalent to ρ . Show that the canonical extension $\tilde{\sigma}$ of σ from $\mathcal{J}$ to $\mathcal{A}$ is unitarily equivalent to $\tilde{\rho}$.