SOLUTIONS

Problem 1.

a) We have $\emptyset \in \mathcal{T}$ and $\mathbb{R} \in \mathcal{T}$ since $\mathbb{Z} \subset \mathbb{R}$.
If $U_\alpha \in \mathcal{T}$ for $\alpha \in J$, then $\mathbb{Z} \subset U_\alpha$ $\forall \alpha$ so $\mathbb{Z} \subset \bigcup_{\alpha \in J} U_\alpha$.

If $U_1, \cdots, U_n \in \mathcal{T}$, then $\mathbb{Z} \subset U_i$ for all i, so $\mathbb{Z} \subset \bigcap_{i=1}^{n} U_i$.

b) If K contains only a finite number of nonintegers, then $K = E \cup \{x_1, \cdots, x_n\}$ with $E \subset \mathbb{Z}$ an $n \geq 0$.
If $\{U_\alpha\}_{\alpha \in J}$ is an open covering of K and $K \neq \emptyset$, then some $U_{\alpha_0} \neq \emptyset$, but then $E \subset \mathbb{Z} \subset U_{\alpha_0}$. If we now pick U_{α_i} such that $x_i \in U_{\alpha_i}$, then $U_{\alpha_0}, \cdots, U_{\alpha_n}$ cover K.
If K contains an infinite set $A = \{x_\alpha\}$ of nonintegers, then the sets $U_\alpha = \mathbb{Z} \cup \{x_\alpha\}$ form an open cover of K which has no finite subcover.

c) The connected sets are 1) Any set that contains at least one integer and 2) Singletons $\{x\}$. If A is a set of type 1) and $k \in A \cap \mathbb{Z}$, then any two nonempty open sets U, V will both contain k, hence cannot be a separation.
Singletons are always connected. If A is not of type 1) or 2), then A contains no integers, but contains at least two nonintegers x_1, x_2. Then $U = \mathbb{Z} \cup \{x_1\}$ and $V = \mathbb{Z} \cup (A \setminus \{x_1\})$ are open sets and $U \cap A = \{x_1\}$ and $V \cap A = A \setminus \{x_1\}$ is a separation of A, which is therefore not connected.

Problem 2

a) A topological space is locally compact if every point has a neighbourhood which is contained in a compact set.

b) Being an open subset of a compact Hausdorff space, Y is a locally compact Hausdorff space and has a one-point compactification $Y^* = Y \cup \{\infty\}$ where the topology is defined by
1) Open sets $U \subset Y$
2) $Y^* \setminus C$, where $C \subset Y$ is compact

We will show that $f: X \to Y^*$ defined by $f(x) = \infty$ and $f(x') = x'$ if $x' \neq x$ is a homeomorphism. We first check continuity: For open sets of type 1), $f^{-1}(U) = U$ which is open in Y and therefore in X. For open sets of type 2), $f^{-1}(Y^* \setminus C) = X \setminus C$ which is open, since C is compact and X is Hausdorff. This proves continuity.

Clearly f is bijective. Since X is compact and Y^* is Hausdorff it follows that f is a homeomorphism.

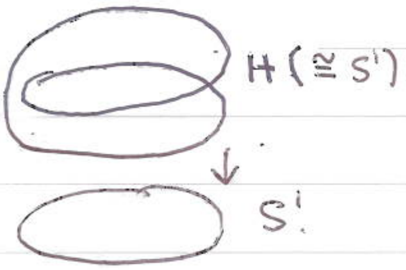
One can also argue that Y^* is homeomorphic to X since the one-point compactification is unique.

c) Each line $\{n\} \times \mathbb{R}$ is homeomorphic to the punctured circle $S_n \setminus \{(0,0)\}$. These homeomorphism f_n define a homeomorphism $f: Z \to Y = X \setminus \{(0,0)\}$ by letting $f = f_n$ on the line $\{n\} \times \mathbb{R}$. The punctured circles are all open subsets of Y and the inverse map $f^{-1}|_{S_n} = f_n^{-1}$ is therefore continuous. Since Z and Y are homeomorphic, their one-point compactifications are homeomorphic. But Y^* is homeomorphic to X by a).

Problem 3

a) For instance $p : S^1 \to S^1$, $p(z) = z^2$.

A figure can for instance be a double helix over S^1



b) No. For every $b_0 \in B$ there is a map $\phi : \pi_1(B, b_0) \to p^{-1}(b$
which is surjective since E is path connected. When $\pi_1(B, b_0)$ is trivial, this means that $p^{-1}(b_0)$ consists of exactly one point. But then $p : E \to B$ is a bijection, hence a homeomorphism.

Problem 4

a) That f is a quotient map means that $B \subset Y$ is closed iff $f^{-1}(B)$ is closed in X.
If f is a closed map and $A \subset X$ is closed, then $f(A)$ is closed, so the saturation $f^{-1}(f(A))$ is closed since f is continuous
Conversely, assume the saturation of closed sets are closed. If A is closed then $f^{-1}(f(A))$ is closed, but since f is a quotient map this means that $f(A)$ is closed in Y.

b) We first show that points are closed in Y. Let $y \in Y$ and assume $x \in f^{-1}(y)$. Then $\{x\}$ is closed, so the saturation $f^{-1}(f(x)) = f^{-1}(y)$ is closed. But then $\{y\}$ is closed since f is a quotient map.

Now, let $B_1, B_2 \subset Y$ be disjoint closed sets. Then $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint closed and saturated sets in X. Hence there are disjoint open sets U_1, U_2 containing A_1 and A_2. Then U_1^c is closed so the saturation $F_1 = f^{-1}(f(U_1^c))$ is a closed set containing U_1^c, but not intersecting A_1, so $V_1 = F_1^c$ is an open saturated set with $A_1 \subset V_1 \subset U_1$. Similarly $V_2 = (f^{-1}(f(U_2^c)))^c$ is an open saturated set with $A_2 \subset V_2 \subset U_2$. But then $W_1 = f(V_1)$ and $W_2 = f(V_2)$ are disjoint open sets containing B_1 and B_2.